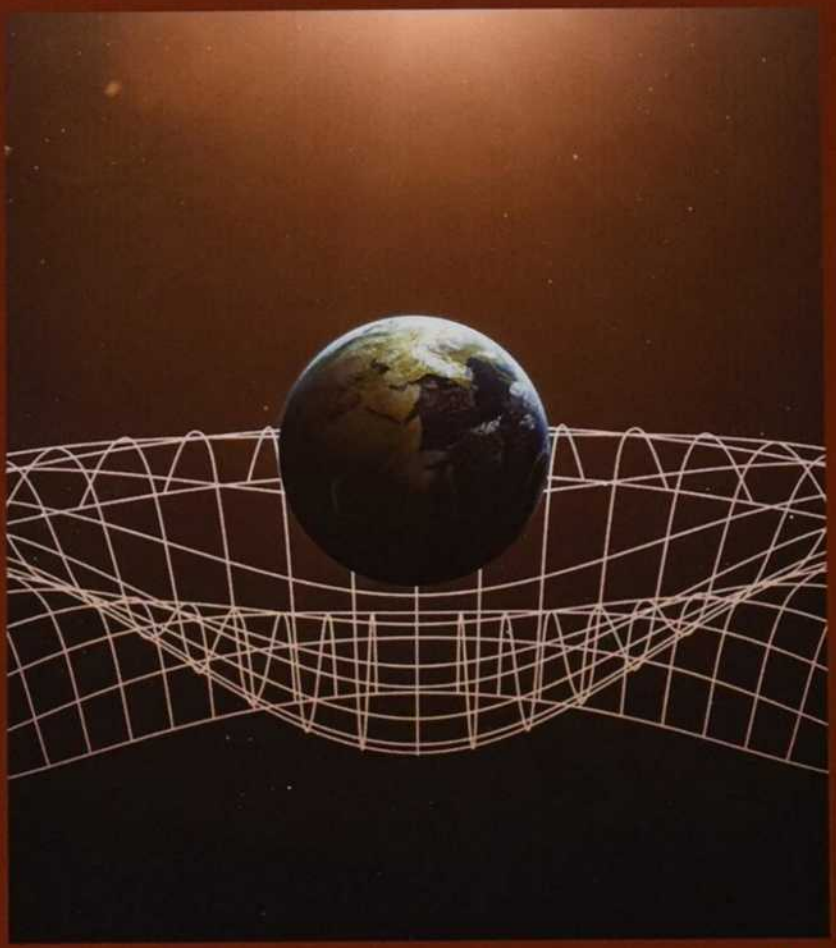
An Introduction to General Relativity

SPACETIME and GEOMETRY



Sean M. Carroll

Spacetime and Geometry

An Introduction to General Relativity

Spacetime and Geometry is an introductory textbook on general relativity, specifically aimed at students. Using a lucid style, Carroll first covers the foundations of the theory and mathematical formalism, providing an approachable introduction to what can often be an intimidating subject. Three major applications of general relativity are then discussed: black holes, perturbation theory and gravitational waves, and cosmology. Students will learn the origin of how spacetime curves (the Einstein equation) and how matter moves through it (the geodesic equation). They will learn what black holes really are, how gravitational waves are generated and detected, and the modern view of the expansion of the universe. A brief introduction to quantum field theory in curved spacetime is also included. A student familiar with this book will be ready to tackle research-level problems in gravitational physics.

Sean M. Carroll is Research Professor of Physics at the California Institute of Technology. His research focuses on general relativity, cosmology, field theory, statistical mechanics, and quantum mechanics. He is the recipient of numerous awards, including the Gemant Award from the American Institute of Physics, the Winton Science Book Prize from the Royal Society, a Guggenheim fellowship, and teaching awards from MIT and the University of Chicago.

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"For if each Star is little more a mathematical Point, located upon the Hemisphere of Heaven by Right Ascension and Declination, then all the Stars, taken together, tho' innumerable, must like any other set of points, in turn represent some single gigantick Equation, to the mind of God as straightforward as, say, the Equation of a Sphere,—to us unreadable, incalculable. A lonely, uncompensated, perhaps even impossible Task,—yet some of us must ever be seeking, I suppose."

—Thomas Pynchon, *Mason & Dixon*

Preface

General relativity is the most beautiful physical theory ever invented. It describes one of the most pervasive features of the world we experience—gravitation—in terms of an elegant mathematical structure—the differential geometry of curved spacetime—leading to unambiguous predictions that have received spectacular experimental confirmation. Consequences of general relativity, from the big bang to black holes, often get young people first interested in physics, and it is an unalloyed joy to finally reach the point in one's studies where these phenomena may be understood at a rigorous quantitative level. If you are contemplating reading this book, that point is here.

In recent decades, general relativity (GR) has become an integral and indispensable part of modern physics. For a long time after it was proposed by Einstein in 1916, GR was counted as a shining achievement that lay somewhat outside the mainstream of interesting research. Increasingly, however, contemporary students in a variety of specialties are finding it necessary to study Einstein's theory. In addition to being an active research area in its own right, GR is part of the standard syllabus for anyone interested in astrophysics, cosmology, string theory, and even particle physics. This is not to slight the more pragmatic uses of GR, including the workings of the Global Positioning System (GPS) satellite network.

There is no shortage of books on GR, and many of them are excellent. Indeed, approximately thirty years ago witnessed the appearance of no fewer than three books in the subject, each of which has become a classic in its own right: those by Weinberg (1972), Misner, Thorne, and Wheeler (1973), and Hawking and Ellis (1975). Each of these books is suffused with a strongly-held point of view advocated by the authors. This has led to a love-hate relationship between these works and their readers; in each case, it takes little effort to find students who will declare them to be the best textbook ever written, or other students who find them completely unpalatable. For the individuals in question, these judgments may very well be correct; there are many different ways to approach this subject.

The present book has a single purpose: to provide a clear introduction to general relativity, suitable for graduate students or advanced undergraduates. I have attempted to include enough material so that almost any one-semester introductory course on GR can find the appropriate subjects covered in the text, but not too much more than that. In particular, I have tried to resist the temptation to write a comprehensive reference book. The only goal of this book is to teach you GR.

An intentional effort has been made to prefer the conventional over the idiosyncratic. If I can be accused of any particular ideological bias, it would be a

tendency to think of general relativity as a field theory, a point of view that helps one to appreciate the connections among GR, particle physics, and string theory. At the same time, there are a number of exciting astrophysical applications of GR (black holes, gravitational lensing, the production and detection of gravitational waves, the early universe, the late universe, the cosmological constant), and I have endeavored to include at least enough background discussion of these issues to prepare students to tackle the current literature.

The primary question facing any introductory treatment of general relativity is the level of mathematical rigor at which to operate. There is no uniquely proper solution, as different students will respond with different levels of understanding and enthusiasm to different approaches. Recognizing this, I have tried to provide something for everyone. I have not shied away from detailed formalism, but have also attempted to include concrete examples and informal discussion of the concepts under consideration. Much of the most mathematical material has been relegated to the Appendices. Some of the material in the Appendices is actually an integral part of the course (for example, the discussion of conformal diagrams), but an individual reader or instructor can decide just when it is appropriate to delve into them; signposts are included in the body of the text.

Surprisingly, there are very few formal prerequisites for learning general relativity; most of the material is developed as we go along. Certainly no prior exposure to Riemannian geometry is assumed, nor would it necessarily be helpful. It would be nice to have already studied some special relativity; although a discussion is included in Chapter 1, its purpose is more to review the basics and and introduce some notation, rather than to provide a self-contained introduction. Beyond that, some exposure to electromagnetism, Lagrangian mechanics, and linear algebra might be useful, but the essentials are included here.

The structure of the book should be clear. The first chapter is a review of special relativity and basic tensor algebra, including a brief discussion of classical field theory. The next two chapters introduce manifolds and curvature in some detail; some motivational physics is included, but building a mathematical framework is the primary goal. General relativity proper is introduced in Chapter 4, along with some discussion of alternative theories. The next four chapters discuss the three major applications of GR: black holes (two chapters), perturbation theory and gravitational waves, and cosmology. Each of these subjects has witnessed an explosion of research in recent years, so the discussions here will be necessarily introductory, but I have tried to emphasize issues of relevance to current work. These three applications can be covered in any order, although there are interdependencies highlighted in the text. Discussions of experimental tests are sprinkled through these chapters. Chapter 9 is a brief introduction to quantum field theory in curved spacetime; this is not a necessary part of a first look at GR, but has become increasingly important to work in quantum gravity and cosmology, and therefore deserves some mention. On the other hand, a few topics are scandalously neglected; the initial-value problem and cosmological perturbation theory come to mind, but there are others. Fortunately there is no shortage of other resources. The Appendices serve various purposes: There are discussions of

technical points that were avoided in the body of the book, crucial concepts that could have been put in various places, and extra topics that are useful but outside the main development.

Since the goal of the book is pedagogy rather than originality, I have often leaned heavily on other books (listed in the bibliography) when their expositions seemed perfectly sensible to me. When this leaning was especially heavy, I have indicated it in the text itself. It will be clear that a primary resource was the book by Wald (1984), which has become a standard reference in the field; readers of this book will hopefully be well-prepared to jump into the more advanced sections of Wald's book.

This book grew out of a set of lecture notes that were prepared when I taught a course on GR at MIT. These notes are available on the web for free, and will continue to be so; they will be linked to the website listed below. Perhaps a little over half of the material here is contained in the notes, although the advantages of owning the book (several copies, even) should go without saying.

Countless people have contributed greatly both to my own understanding of general relativity and to this book in particular—too many to acknowledge with any hope of completeness. Some people, however, deserve special mention. Ted Pyne learned the subject along with me, taught me a great deal, and collaborated with me the first time we taught a GR course, as a seminar in the astronomy department at Harvard; parts of this book are based on our mutual notes. Nick Warner taught the course at MIT from which I first learned GR, and his lectures were certainly a very heavy influence on what appears here. Neil Cornish was kind enough to provide a wealth of exercises, many of which have been included at the end of each chapter. And among the many people who have read parts of the manuscript and offered suggestions, Sanaz Arkani-Hamed was kind enough to go through the entire thing in great detail.

I would also like to thank everyone who either commented in person or by email on different parts of the book; these include Tigran Aivazian, Teodora Beloreshka, Ed Bertschinger, Patrick Brady, Peter Brown, Jennifer Chen, Michele Ferraz Figueiró, Eanna Flanagan, Jacques Fric, Ygor Geurts, Marco Godina, Monica Guica, Jim Hartle, Tamás Hauer, Daniel Holz, Ted Jacobson, Akash Kansagra, Chuck Keeton, Arthur Kosowsky, Eugene Lim, Jorma Louko, Robert A. McNees, Hayri Mutluay, Simon Ross, Itai Seggev, Robert Wald, and Barton Zwiebach. Apologies are due to anyone I may have neglected to mention. And along the way I was fortunate to be the recipient of wisdom and perspective from numerous people, including Shadi Bartsch, George Field, Deryn Fogg, Ilana Harrus, Gretchen Helfrich, Mari Ruti, Maria Spiropulu, Mark Trodden, and of course my family. (This wisdom often came in the form, "What were you thinking?") Finally, I would like to thank the students in my GR classes, on whom the strategies deployed here were first tested, and express my gratitude to my students and collaborators, for excusing my book-related absences when I should have been doing research.

My friends who have written textbooks themselves tell me that the first printing of a book will sometimes contain mistakes. In the unlikely event that this happens

here, there will be a list of errata kept at the website for the book:

`http://spacetimeandgeometry.net/`

The website will also contain other relevant links of interest to readers.

During the time I was working on this book, I was supported by the National Science Foundation, the Department of Energy, the Alfred P. Sloan Foundation, and the David and Lucile Packard Foundation.

Sean Carroll
Chicago, Illinois
June 2003

Contents

CHAPTER

1 Special Relativity and Flat Spacetime

1.1 ■ PRELUDE

General relativity (GR) is Einstein's theory of space, time, and gravitation. At heart it is a very simple subject (compared, for example, to anything involving quantum mechanics). The essential idea is perfectly straightforward: while most forces of nature are represented by fields defined on spacetime (such as the electromagnetic field, or the short-range fields characteristic of subnuclear forces), gravity is inherent in spacetime itself. In particular, what we experience as "gravity" is a manifestation of the *curvature* of spacetime.

Our task, then, is clear. We need to understand spacetime, we need to understand curvature, and we need to understand how curvature becomes gravity. Roughly, the first two chapters of this book are devoted to an exploration of spacetime, the third is about curvature, and the fourth explains the relationship between curvature and gravity, before we get into applications of the theory. However, let's indulge ourselves with a short preview of what is to come, which will perhaps motivate the initial steps of our journey.

GR is a theory of gravity, so we can begin by remembering our previous theory of gravity, that of Newton. There are two basic elements: an equation for the gravitational field as influenced by matter, and an equation for the response of matter to this field. The conventional Newtonian statement of these rules is in terms of forces between particles; the force between two objects of masses M and m separated by a vector $\mathbf{r} = r\mathbf{e}_{(r)}$ is the famous inverse-square law,

$$\mathbf{F} = -\frac{GMm}{r^2}\mathbf{e}_{(r)}, \tag{1.1}$$

and this force acts on a particle of mass m to give it an acceleration according to Newton's second law,

$$\mathbf{F} = m\mathbf{a}. \tag{1.2}$$

Equivalently, we could use the language of the gravitational potential Φ; the potential is related to the mass density ρ by Poisson's equation,

$$\nabla^2\Phi = 4\pi G\rho, \tag{1.3}$$

and the acceleration is given by the gradient of the potential,

$$\mathbf{a} = -\nabla\Phi. \tag{1.4}$$

Either (1.1) and (1.2), or (1.3) and (1.4), serve to define Newtonian gravity. To define GR, we need to replace each of them by statements about the curvature of spacetime.

The hard part is the equation governing the response of spacetime curvature to the presence of matter and energy. We will eventually find what we want in the form of Einstein's equation,

$$R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{1.5}$$

This looks more forbidding than it should, largely because of those Greek subscripts. In fact this is simply an equation between 4×4 matrices, and the subscripts label elements of each matrix. The expression on the left-hand side is a measure of the curvature of spacetime, while the right-hand side measures the energy and momentum of matter, so this equation relates energy to curvature, as promised. But we will defer until later a detailed understanding of the inner workings of Einstein's equation.

The response of matter to spacetime curvature is somewhat easier to grasp: Free particles move along paths of "shortest possible distance," or geodesics. In other words, particles try their best to move on straight lines, but in a curved spacetime there might not be any straight lines (in the sense we are familiar with from Euclidean geometry), so they do the next best thing. Their parameterized paths $x^\mu(\lambda)$ obey the geodesic equation:

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda} = 0. \tag{1.6}$$

At this point you aren't expected to understand (1.6) any more than (1.5); but soon enough it will all make sense.

As we will discuss later, the universal nature of geodesic motion is an extremely profound feature of GR. This universality is the origin of our claim that gravity is not actually a "force," but a feature of spacetime. A charged particle in an electric field feels an acceleration, which deflects it from straight-line motion; in contrast, a particle in a gravitational field moves along a path that is the closest thing there is to a straight line. Such particles do not feel acceleration; they are freely falling. Once we become more familiar with the spirit of GR, it will make perfect sense to think of a ball flying through the air as being more truly "unaccelerated" than one sitting on a table; the one sitting on a table is being deflected away from the geodesic it would like to be on (which is why we feel a force on our feet as we stand on Earth).

The basic concept underlying our description of spacetime curvature will be that of the metric tensor, typically denoted by $g_{\mu\nu}$. The metric encodes the geometry of a space by expressing deviations from Pythagoras's theorem, $(\Delta l)^2 = (\Delta x)^2 + (\Delta y)^2$ (where Δl is the distance between two points defined on a Cartesian grid with coordinate separations Δx and Δy). This familiar formula is valid only in conventional Euclidean geometry, where it is implicitly assumed that space is flat. In the presence of curvature our deeply ingrained notions of ge-

ometry will begin to fail, and we can characterize the amount of curvature by keeping track of how Pythagoras's relation is altered. This information is contained in the metric tensor. From the metric we will derive the Riemann curvature tensor, used to define Einstein's equation, and also the geodesic equation. Setting up this mathematical apparatus is the subject of the next several chapters.

Despite the need to introduce a certain amount of formalism to discuss curvature in a quantitative way, the essential notion of GR ("gravity is the curvature of spacetime") is quite simple. So why does GR have, at least in some benighted circles, a reputation for difficulty or even abstruseness? Because the elegant truths of Einstein's theory are obscured by the accumulation of certain pre-relativity notions which, although very useful, must first be discarded in order to appreciate the world according to GR. Specifically, we live in a world in which spacetime curvature is very small, and particles are for the most part moving quite slowly compared to the speed of light. Consequently, the mechanics of Galileo and Newton comes very naturally to us, even though it is only an approximation to the deeper story.

So we will set about learning the deeper story by gradually stripping away the layers of useful but misleading Newtonian intuition. The first step, which is the subject of this chapter, will be to explore special relativity (SR), the theory of spacetime in the absence of gravity (curvature). Hopefully this is mostly review, as it will proceed somewhat rapidly. The point will be both to recall what SR is all about, and to introduce tensors and related concepts that will be crucial later on, without the extra complications of curvature on top of everything else. Therefore, for this chapter we will always be working in flat spacetime, and furthermore we will only use inertial (Cartesian-like) coordinates. Needless to say it is possible to do SR in any coordinate system you like, but it turns out that introducing the necessary tools for doing so would take us halfway to curved spaces anyway, so we will put that off for a while.

1.2 ■ SPACE AND TIME, SEPARATELY AND TOGETHER

A purely cold-blooded approach to GR would reverse the order of Chapter 2 (Manifolds) and Chapter 1 (Special Relativity and Flat Spacetime). A *manifold* is the kind of mathematical structure used to describe spacetime, while *special relativity* is a model that invokes a particular kind of spacetime (one with no curvature, and hence no gravity). However, if you are reading this book you presumably have at least some familiarity with special relativity (SR), while you may not know anything about manifolds. So our first step will be to explore the relatively familiar territory of SR, taking advantage of this opportunity to introduce concepts and notation that will be crucial to later developments.

Special relativity is a theory of the structure of spacetime, the background on which particles and fields evolve. SR serves as a replacement for Newtonian mechanics, which also is a theory of the structure of spacetime. In either case, we can distinguish between this basic structure and the various dynamical laws govern-

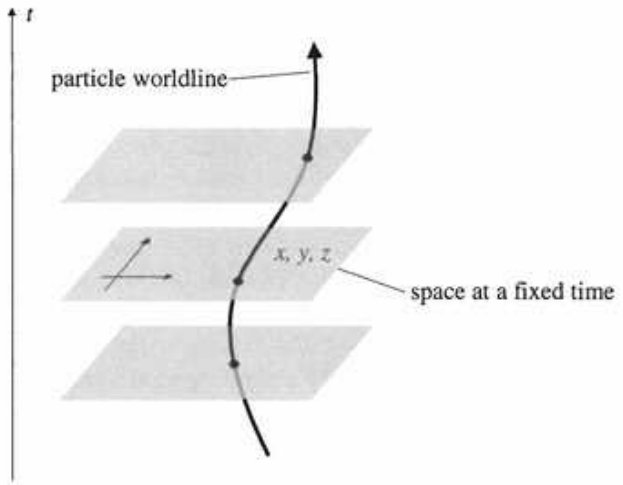


FIGURE 1.1 In Newtonian spacetime there is an absolute slicing into distinct copies of space at different moments in time. Particle worldlines are constrained to move forward in time, but can travel through space at any velocity; there is universal agreement on the question of whether two events at different points in space occur at the same moment of time.

ing specific systems: Newtonian gravity is an example of a dynamical system set within the context of Newtonian mechanics, while Maxwell's electromagnetism is a dynamical system operating within the context of special relativity.

Spacetime is a four-dimensional set, with elements labeled by three dimensions of space and one of time. (We'll do a more rigorous job with the definitions in the next chapter.) An individual point in spacetime is called an **event**. The path of a particle is a curve through spacetime, a parameterized one-dimensional set of events, called the **worldline**. Such a description applies equally to SR and Newtonian mechanics. In either case, it seems clear that "time" is treated somewhat differently than "space"; in particular, particles always travel forward in time, whereas they are free to move back and forth in space.

There is an important difference, however, between the set of allowed paths that particles can take in SR and those in Newton's theory. In Newtonian mechanics, there is a basic division of spacetime into well-defined slices of "all of space at a fixed moment in time." The notion of *simultaneity*, when two events occur at the same time, is unambiguously defined. Trajectories of particles will move ever forward in time, but are otherwise unconstrained; in particular, there is no limit on the relative velocity of two such particles.

In SR the situation is dramatically altered: in particular, *there is no well-defined notion of two separated events occurring "at the same time."* That is not to say that spacetime is completely structureless. Rather, at any event we can define a **light cone**, which is the locus of paths through spacetime that could conceivably be taken by light rays passing through this event. The absolute division, in Newtonian

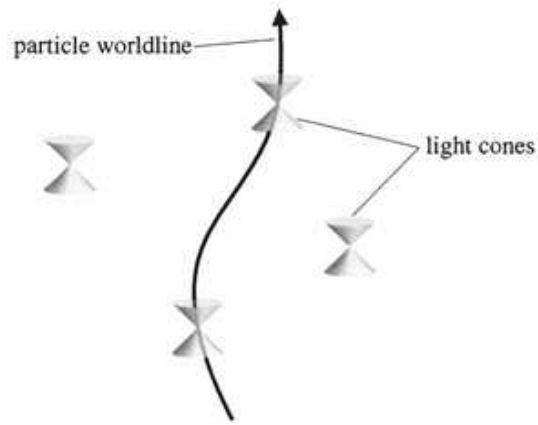


FIGURE 1.2 In special relativity there is no absolute notion of "all of space at one moment in time." Instead, there is a rule that particles always travel at less than or equal to the speed of light. We can therefore define light cones at every event, which locally describe the set of allowed trajectories. For two events that are outside each others' light cones, there is no universal notion of which event occurred earlier in time.

mechanics, of spacetime into unique slices of space parameterized by time, is replaced by a rule that says that physical particles cannot travel faster than light, and consequently move along paths that always remain inside these light cones.

The absence of a preferred time-slicing in SR is at the heart of why the notion of spacetime is more fundamental in this context than in Newtonian mechanics. Of course we can choose specific coordinate systems in spacetime, and once we do, it makes sense to speak of separated events occurring at the same value of the time coordinate in this particular system; but there will also be other possible coordinates, related to the first by "rotating" space and time into each other. This phenomenon is a natural generalization of rotations in Euclidean geometry, to which we now turn.

Consider a garden-variety two-dimensional plane. It is typically convenient to label the points on such a plane by introducing coordinates, for example by defining orthogonal x and y axes and projecting each point onto these axes in the usual way. However, it is clear that most of the interesting geometrical facts about the plane are independent of our choice of coordinates; there aren't any preferred directions. As a simple example, we can consider the distance between two points, given by

$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2. \tag{1.7}$$

In a different Cartesian coordinate system, defined by x' and y' axes that are rotated with respect to the originals, the formula for the distance is unaltered:

$$(\Delta s)^2 = (\Delta x')^2 + (\Delta y')^2. \tag{1.8}$$

We therefore say that the distance is invariant under such changes of coordinates.

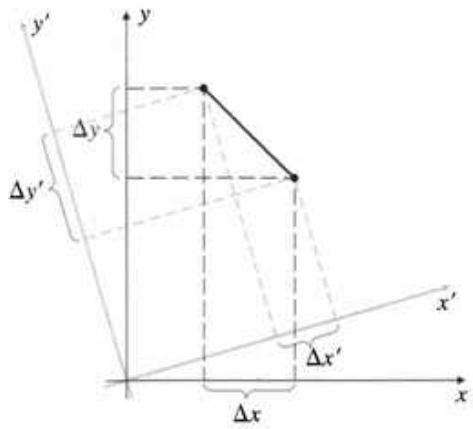


FIGURE 1.3 Two-dimensional Euclidean space, with two different coordinate systems. Notions such as "the distance between two points" are independent of the coordinate system chosen.

This is why it is useful to think of the plane as an intrinsically two-dimensional space, rather than as two fundamentally distinct one-dimensional spaces brought arbitrarily together: Although we use two distinct numbers to label each point, the numbers are not the essence of the geometry, since we can rotate axes into each other while leaving distances unchanged. In Newtonian physics this is not the case with space and time; there is no useful notion of rotating space and time into each other. Rather, the notion of "all of space at a single moment in time" has a meaning independent of coordinates.

SR is a different story. Let us consider coordinates (t, x, y, z) on spacetime, set up in the following way. The spatial coordinates (x, y, z) comprise a standard Cartesian system, constructed for example by welding together rigid rods that meet at right angles. The rods must be moving freely, unaccelerated. The time coordinate is defined by a set of clocks, which are not moving with respect to the spatial coordinates. (Since this is a thought experiment, we can imagine that the rods are infinitely long and there is one clock at every point in space.) The clocks are synchronized in the following sense. Imagine that we send a beam of light from point 1 in space to point 2, in a straight line at a constant velocity c, and then immediately back to 1 (at velocity $-c$). Then the time on the coordinate clock when the light beam reaches point 2, which we label t_2, should be halfway between the time on the coordinate clock when the beam left point 1 (t_1) and the time on that same clock when it returned (t_1'):

$$t_2 = \tfrac{1}{2}(t_1' + t_1). \tag{1.9}$$

The coordinate system thus constructed is an **inertial frame**, or simply "inertial coordinates." These coordinates are the natural generalization to spacetime of Cartesian (orthonormal) coordinates in space. (The reason behind the careful

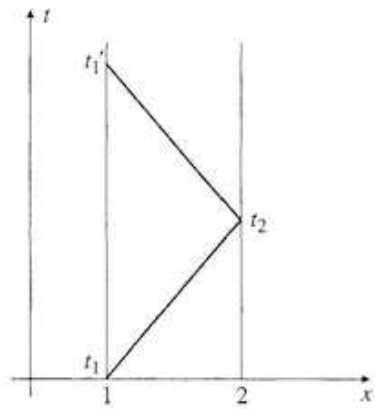


FIGURE 1.4 Synchronizing clocks in an inertial coordinate system. The clocks are synchronized if the time t_2 is halfway between t_1 and t_1' when we bounce a beam of light from point 1 to point 2 and back.

construction is so that we only make comparisons *locally*; never, for example, comparing two far-away clocks to each other at the same time. This kind of care will be even more necessary once we go to general relativity, where there will not be any way to construct inertial coordinates throughout spacetime.)

We can construct any number of inertial frames via this procedure, differing from the first one by an offset in initial position and time, angle, and (constant) velocity. In a Newtonian world, the new coordinates (t', x', y', z') would have the feature that $t' = t +$ constant, independent of spatial coordinates. That is, there is an absolute notion of "two events occurring simultaneously, that is, at the same time." But in SR this isn't true; in general the three-dimensional "spaces" defined by $t =$ constant will differ from those defined by $t' =$ constant.

However, we have not descended completely into chaos. Consider, without any motivation for the moment, what we will call the **spacetime interval** between two events:

$$\boxed{(\Delta s)^2 = -(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2.} \tag{1.10}$$

(Notice that it can be positive, negative, or zero even for two nonidentical points.) Here, c is some fixed conversion factor between space and time, that is, a fixed velocity. As an empirical matter, it turns out that electromagnetic waves propagate in vacuum at this velocity c, which we therefore refer to as "the speed of light." The important thing, however, is not that photons happen to travel at that speed, but that there exists a c such that *the spacetime interval is invariant under changes of inertial coordinates*. In other words, if we set up a new inertial frame (t', x', y', z'), the interval will be of the same form:

$$(\Delta s)^2 = -(c\Delta t')^2 + (\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2. \tag{1.11}$$

This is why it makes sense to think of SR as a theory of four-dimensional spacetime, known as **Minkowski space**. (This is a special case of a four-dimensional manifold, which we will deal with in detail later.) As we shall see, the coordinate transformations that we have implicitly defined do, in a sense, rotate space and time into each other. There is no absolute notion of "simultaneous events"; whether two things occur at the same time depends on the coordinates used. Therefore, the division of Minkowski space into space and time is a choice we make for our own purposes, not something intrinsic to the situation.

Almost all of the "paradoxes" associated with SR result from a stubborn persistence of the Newtonian notions of a unique time coordinate and the existence of "space at a single moment in time." By thinking in terms of spacetime rather than space and time together, these paradoxes tend to disappear.

Let's introduce some convenient notation. Coordinates on spacetime will be denoted by letters with Greek superscript indices running from 0 to 3, with 0 generally denoting the time coordinate. Thus,

$$x^\mu : \begin{array}{l} x^0 = ct \\ x^1 = x \\ x^2 = y \\ x^3 = z. \end{array} \tag{1.12}$$

(Don't start thinking of the superscripts as exponents.) Furthermore, for the sake of simplicity we will choose units in which

$$c = 1; \tag{1.13}$$

we will therefore leave out factors of c in all subsequent formulae. Empirically we know that c is 3×10^8 meters per second; thus, we are working in units where 1 second equals 3×10^8 meters. Sometimes it will be useful to refer to the space and time components of x^μ separately, so we will use Latin superscripts to stand for the space components alone:

$$x^i : \begin{array}{l} x^1 = x \\ x^2 = y \\ x^3 = z. \end{array} \tag{1.14}$$

It is also convenient to write the spacetime interval in a more compact form. We therefore introduce a 4×4 matrix, the **metric**, which we write using two lower indices:

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.15}$$

(Some references, especially field theory books, define the metric with the opposite sign, so be careful.) We then have the nice formula

$$(\Delta s)^2 = \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu. \tag{1.16}$$

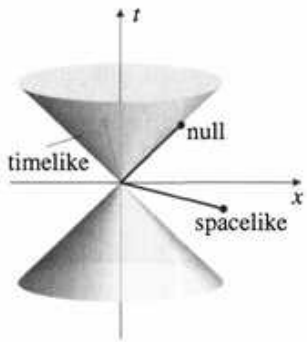


FIGURE 1.5 A light cone, portrayed on a spacetime diagram. Points that are spacelike-, null-, and timelike-separated from the origin are indicated.

This formula introduces the **summation convention**, in which indices appearing both as superscripts and subscripts are summed over. We call such labels **dummy indices**; it is important to remember that they are summed over all possible values, rather than taking any specific one. (It will always turn out to be the case that dummy indices occur strictly in pairs, with one "upstairs" and one "downstairs." More on this later.) The content of (1.16) is therefore exactly the same as (1.10).

An extremely useful tool is the **spacetime diagram**, so let's consider Minkowski space from this point of view. We can begin by portraying the initial t and x axes at right angles, and suppressing the y and z axes. ("Right angles" as drawn on a spacetime diagram don't necessarily imply "orthogonal in spacetime," although that turns out to be true for the t and x axes in this case.) It is enlightening to consider the paths corresponding to travel at the speed $c = 1$, given by $x = \pm t$. A set of points that are all connected to a single event by straight lines moving at the speed of light is the **light cone**, since if we imagine including one more spatial coordinate, the two diagonal lines get completed into a cone. Light cones are naturally divided into future and past; the set of all points inside the future and past light cones of a point p are called **timelike separated** from p, while those outside the light cones are **spacelike separated** and those on the cones are **lightlike** or **null separated** from p. Referring back to (1.10), we see that the interval between timelike separated points is negative, between spacelike separated points is positive, and between null separated points is zero. (The interval is defined to be $(\Delta s)^2$, not the square root of this quantity.)

The fact that the interval is negative for a timelike line (on which a slower-than-light particle will actually move) is annoying, so we define the **proper time** τ to satisfy

$$(\Delta\tau)^2 = -(\Delta s)^2 = -\eta_{\mu\nu}\Delta x^\mu \Delta x^\nu. \tag{1.17}$$

A crucial feature of the spacetime interval is that *the proper time between two events measures the time elapsed as seen by an observer moving on a straight path between the events*. This is easily seen in the very special case that the two events have the same spatial coordinates, and are only separated in time; this corresponds to the observer traveling between the events being at rest in the coordinate system used. Then $(\Delta\tau)^2 = -\eta_{\mu\nu}\Delta x^\mu \Delta x^\nu = (\Delta t)^2$, so $\Delta\tau = \Delta t$, and of course we defined t as the time measured by a clock located at a fixed spatial position. But the spacetime interval is invariant under changes of inertial frame; the proper time (1.17) between two fixed events will be the same when evaluated in an inertial frame where the observer is moving as it is in the frame where the observer is at rest.

A crucial fact is that, for more general trajectories, the proper time and coordinate time are different (although the proper time is always that measured by the clock carried by an observer along the trajectory). Consider two trajectories between events A and C, one a straight line passing through a halfway point marked B, and another traveled by an observer moving away from A at a constant velocity $v = dx/dt$ to a point B' and then back at a constant velocity $-v$ to intersect at

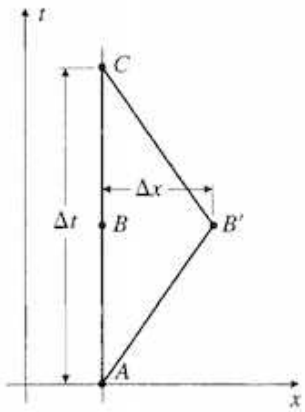


FIGURE 1.6 The twin paradox. A traveler on the straight path through spacetime ABC will age more than someone on the nonstraight path $AB'C$. Since proper time is a measure of distance traveled through spacetime, this should come as no surprise. (The only surprise might be that the straight path is the one of *maximum* proper time; this can be traced to the minus sign for the timelike component of the metric.)

the event C. Choose inertial coordinates such that the straight trajectory describes a motionless particle, with event A located at coordinates $(t, x) = (0, 0)$ and C located at $(\Delta t, 0)$. The two paths then describe an isosceles triangle in spacetime; B has coordinates $(\frac{1}{2}\Delta t, 0)$ and B' has coordinates $(\frac{1}{2}\Delta t, \Delta x)$, with $\Delta x = \frac{1}{2}v\Delta t$. Clearly, $\Delta\tau_{AB} = \frac{1}{2}\Delta t$, but

$$\begin{aligned}\Delta\tau_{AB'} &= \sqrt{(\tfrac{1}{2}\Delta t)^2 - (\Delta x)^2} \\ &= \tfrac{1}{2}\sqrt{1 - v^2}\,\Delta t. \end{aligned} \tag{1.18}$$

It should be obvious that $\Delta\tau_{BC} = \Delta\tau_{AB}$ and $\Delta\tau_{B'C} = \Delta\tau_{AB'}$. Thus, the observer on the straight-line trip from event A to C experiences an elapsed time of $\Delta\tau_{ABC} = \Delta t$, whereas the one who traveled out and returned experiences

$$\Delta\tau_{AB'C} = \sqrt{1 - v^2}\,\Delta t < \Delta t. \tag{1.19}$$

Even though the two observers begin and end at the same points in spacetime, they have aged different amounts. This is the famous "twin paradox," the unfortunate scene of all sorts of misunderstandings and tortured explanations. The truth is straightforward: a nonstraight path in spacetime has a different interval than a straight path, just as a nonstraight path in space has a different length than a straight one. This isn't as trivial as it sounds, of course; the profound insight is the way in which "elapsed time along a worldline" is related to the interval traversed through spacetime. In a Newtonian world, the coordinate t represents a universal flow of time throughout all of spacetime; in relativity, t is just a convenient coordinate, and the elapsed time depends on the path along which you travel. An

important distinction is that the nonstraight path has a *shorter* proper time. In space, the shortest distance between two points is a straight line; in spacetime, the longest proper time between two events is a straight trajectory.

Not all trajectories are nice enough to be constructed from pieces of straight lines. In more general circumstances it is useful to introduce the infinitesimal interval, or **line element**:

$$\boxed{ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu,} \tag{1.20}$$

for infinitesimal coordinate displacements dx^μ. (We are being quite informal here, but we'll make amends later on.) From this definition it is tempting to take the square root and integrate along a path to obtain a finite interval, but it is somewhat unclear what $\int \sqrt{\eta_{\mu\nu} dx^\mu dx^\nu}$ is supposed to mean. Instead we consider a path through spacetime as a parameterized curve, $x^\mu(\lambda)$. Note that, unlike conventional practice in Newtonian mechanics, the parameter λ is not necessarily identified with the time coordinate. We can then calculate the derivatives $dx^\mu/d\lambda$, and write the path length along a spacelike curve (one whose infinitesimal intervals are spacelike) as

$$\Delta s = \int \sqrt{\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}\, d\lambda, \tag{1.21}$$

where the integral is taken over the path. For timelike paths we use the proper time

$$\Delta \tau = \int \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}\, d\lambda, \tag{1.22}$$

which will be positive. (For null paths the interval is simply zero.) Of course we may consider paths that are timelike in some places and spacelike in others, but fortunately it is seldom necessary since the paths of physical particles never change their character (massive particles move on timelike paths, massless particles move on null paths). Once again, $\Delta\tau$ really is the time measured by an observer moving along the trajectory.

The notion of *acceleration* in special relativity has a bad reputation, for no good reason. Of course we were careful, in setting up inertial coordinates, to make sure that particles at rest in such coordinates are unaccelerated. However, once we've set up such coordinates, we are free to consider any sort of trajectories for physical particles, whether accelerated or not. In particular, there is no truth to the rumor that SR is unable to deal with accelerated trajectories, and general relativity must be invoked. General relativity becomes relevant in the presence of gravity, when spacetime becomes curved. Any processes in flat spacetime are described within the context of special relativity; in particular, expressions such as (1.22) are perfectly general.

1.3 ■ LORENTZ TRANSFORMATIONS

We can now consider coordinate transformations in spacetime at a somewhat more abstract level than before. We are interested in a formal description of how to relate the various inertial frames constructed via the procedure outlined above; that is, coordinate systems that leave the interval (1.16) invariant. One simple variety are the **translations**, which merely shift the coordinates (in space or time):

$$x^\mu \rightarrow x^{\mu'} = \delta^{\mu'}_{\mu}(x^\mu + a^\mu), \tag{1.23}$$

where a^μ is a set of four fixed numbers and $\delta^{\mu'}_{\mu}$ is the four-dimensional version of the traditional Kronecker delta symbol:

$$\delta^{\mu'}_{\mu} = \begin{cases} 1 & \text{when } \mu' = \mu, \\ 0 & \text{when } \mu' \neq \mu. \end{cases} \tag{1.24}$$

Notice that we put the prime on the index, not on the x. The reason for this should become more clear once we start dealing with vectors and tensors; the notation serves to remind us that the geometrical object is the same, but its components are resolved with respect to a different coordinate system. Translations leave the differences Δx^μ unchanged, so it is not remarkable that the interval is unchanged. The other relevant transformations include spatial **rotations** and offsets by a constant velocity vector, or **boosts**; these are linear transformations, described by multiplying x^μ by a (spacetime-independent) matrix:

$$x^{\mu'} = \Lambda^{\mu'}{}_{\nu} x^\nu, \tag{1.25}$$

or, in more conventional matrix notation,

$$x' = \Lambda x. \tag{1.26}$$

(We will generally use indices, rather than matrix notation, but right now we have an interest in relating our discussion to certain other familiar notions usually described by matrices.) These transformations do not leave the differences Δx^μ unchanged, but multiply them also by the matrix Λ. What kind of matrices will leave the interval invariant? Sticking with the matrix notation, what we would like is

$$\begin{aligned} (\Delta s)^2 = (\Delta x)^{\mathrm{T}} \eta (\Delta x) &= (\Delta x')^{\mathrm{T}} \eta (\Delta x') \\ &= (\Delta x)^{\mathrm{T}} \Lambda^{\mathrm{T}} \eta \Lambda (\Delta x), \end{aligned} \tag{1.27}$$

and therefore

$$\eta = \Lambda^{\mathrm{T}} \eta \Lambda, \tag{1.28}$$

or

$$\eta_{\rho\sigma} = \Lambda^{\mu'}{}_{\rho}\eta_{\mu'\nu'}\Lambda^{\nu'}{}_{\sigma} = \Lambda^{\mu'}{}_{\rho}\Lambda^{\nu'}{}_{\sigma}\eta_{\mu'\nu'}. \tag{1.29}$$

(In matrix notation the order matters, while in index notation it is irrelevant.) We want to find the matrices $\Lambda^{\mu'}{}_{\nu}$ such that the components of the matrix $\eta_{\mu'\nu'}$ are the same as those of $\eta_{\rho\sigma}$; that is what it means for the interval to be invariant under these transformations.

The matrices that satisfy (1.28) are known as the **Lorentz transformations**; the set of them forms a group under matrix multiplication, known as the **Lorentz group**. There is a close analogy between this group and SO(3), the rotation group in three-dimensional space. The rotation group can be thought of as 3×3 matrices R that satisfy $R^{\mathrm{T}}R = \mathbf{1}$, where $\mathbf{1}$ is the 3×3 identity matrix. Such matrices are called *orthogonal*, and the 3×3 ones form the group O(3). This includes not only rotations but also reversals of orientation of the spatial axes (parity transformations). Sometimes we choose to exclude parity transformations by also demanding that the matrices have unit determinant, $|R| = 1$; such matrices are called *special*, and the resulting group is SO(3). The orthogonality condition can be made to look more like (1.28) if we write it as

$$\mathbf{1} = R^{\mathrm{T}}\mathbf{1}R. \tag{1.30}$$

So the difference between the rotation group O(3) and the Lorentz group is the replacement of $\mathbf{1}$, a 3×3 diagonal matrix with all entries equal to $+1$, by η, a 4×4 diagonal matrix with one entry equal to -1 and the rest equal to $+1$. The Lorentz group is therefore often referred to as O(3,1). It includes not only boosts and rotations, but discrete reversals of the time direction as well as parity transformations. As before we can demand that $|\Lambda| = 1$, leaving the "proper Lorentz group" SO(3,1). However, this does not leave us with what we really want, which is the set of continuous Lorentz transformations (those connected smoothly to the identity), since a combination of a time reversal and a parity reversal would have unit determinant. From the $(\rho, \sigma) = (0, 0)$ component of (1.29) we can easily show that $|\Lambda^{0'}{}_{0}| \geq 1$, with negative values corresponding to time reversals. We can therefore demand at last that $\Lambda^{0'}{}_{0} \geq 1$ (in addition to $|\Lambda| = 1$), leaving the "proper orthochronous" or "restricted" Lorentz group. Sometimes this is denoted by something like $\mathrm{SO}(3,1)^{\uparrow}$, but usually we will not bother to make this distinction explicitly. Note that the 3×3 identity matrix is simply the metric for ordinary flat space. Such a metric, in which all of the eigenvalues are positive, is called **Euclidean**, while those such as (1.15), which feature a single minus sign, are called **Lorentzian**.

It is straightforward to write down explicit expressions for simple Lorentz transformations. A familiar rotation in the x-y plane is:

$$\Lambda^{\mu'}{}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.31}$$

The rotation angle θ is a periodic variable with period 2π. The boosts may be thought of as "rotations between space and time directions." An example is given by a boost in the x-direction:

$$\Lambda^{\mu'}{}_{\nu} = \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.32}$$

The boost parameter ϕ, unlike the rotation angle, is defined from $-\infty$ to ∞. A general transformation can be obtained by multiplying the individual transformations; the explicit expression for this six-parameter matrix (three boosts, three rotations) is not pretty, or sufficiently useful to bother writing down. In general Lorentz transformations will not commute, so the Lorentz group is nonabelian. The set of both translations and Lorentz transformations is a ten-parameter nonabelian group, the **Poincaré group**.

You should not be surprised to learn that the boosts correspond to changing coordinates by moving to a frame that travels at a constant velocity, but let's see it more explicitly. (Don't confuse "boosting" with "accelerating." The difference between boosting to a different reference frame and accelerating an object is the same as the difference between rotating to a different coordinate system and setting an object spinning.) For the transformation given by (1.32), the transformed coordinates t' and x' will be given by

$$\begin{aligned} t' &= t\cosh\phi - x\sinh\phi \\ x' &= -t\sinh\phi + x\cosh\phi. \end{aligned} \tag{1.33}$$

From this we see that the point defined by $x' = 0$ is moving; it has a velocity

$$v = \frac{x}{t} = \frac{\sinh\phi}{\cosh\phi} = \tanh\phi. \tag{1.34}$$

To translate into more pedestrian notation, we can replace $\phi = \tanh^{-1} v$ to obtain

$$\begin{aligned} t' &= \gamma(t - vx) \\ x' &= \gamma(x - vt), \end{aligned} \tag{1.35}$$

where $\gamma = 1/\sqrt{1-v^2}$. So indeed, our abstract approach has recovered the conventional expressions for Lorentz transformations. Applying these formulae leads to time dilation, length contraction, and so forth.

It's illuminating to consider Lorentz transformations in the context of spacetime diagrams. According to (1.33), under a boost in the x-t plane the x' axis ($t' = 0$) is given by $t = x\tanh\phi$, while the t' axis ($x' = 0$) is given by $t = x/\tanh\phi$. We therefore see that the space and time axes are rotated into each other, although they scissor together instead of remaining orthogonal in the traditional Euclidean sense. (As we shall see, the axes do in fact remain orthogonal in

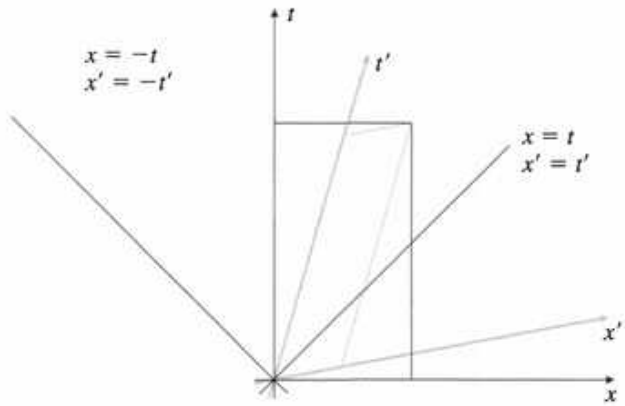


FIGURE 1.7 A Lorentz transformation relates the $\{t', x'\}$ coordinates to the $\{t, x\}$ coordinates. Note that light cones are unchanged.

the Lorentzian sense; that's the implication of the metric remaining invariant under boosts.) This should come as no surprise, since if spacetime behaved just like a four-dimensional version of space the world would be a very different place. We see quite vividly the distinction between this situation and the Newtonian world; in SR, it is impossible to say (in a coordinate-independent way) whether a point that is spacelike separated from p is in the future of p, the past of p, or "at the same time."

Note also that the paths defined by $x' = \pm t'$ are precisely the same as those defined by $x = \pm t$; these trajectories are left invariant under boosts along the x-axis. Of course we know that light travels at this speed; we have therefore found that the speed of light is the same in any inertial frame.

1.4 ■ VECTORS

To probe the structure of Minkowski space in more detail, it is necessary to introduce the concepts of vectors and tensors. We will start with vectors, which should be familiar. Of course, in spacetime vectors are four-dimensional, and are often referred to as **four-vectors**. This turns out to make quite a bit of difference—for example, there is no such thing as a cross product between two four-vectors.

Beyond the simple fact of dimensionality, the most important thing to emphasize is that each vector is located at a given point in spacetime. You may be used to thinking of vectors as stretching from one point to another in space, and even of "free" vectors that you can slide carelessly from point to point. These are not useful concepts outside the context of flat spaces; once we introduce curvature, we lose the ability to draw preferred curves from one point to another, or to move vectors uniquely around a manifold. Rather, to each point p in spacetime we associate the set of all possible vectors located at that point; this set is known as the **tangent space** at p, or T_p. The name is inspired by thinking of the set of

vectors attached to a point on a simple curved two-dimensional space as comprising a plane tangent to the point. (This picture relies on an embedding of the manifold and the tangent space in a higher-dimensional external space, which we won't generally have or need.) Inspiration aside, it is important to think of these vectors as being located at a single point, rather than stretching from one point to another (although this won't stop us from drawing them as arrows on spacetime diagrams).

In Chapter 2 we will relate the tangent space at each point to things we can construct from the spacetime itself. For right now, just think of T_p as an abstract vector space for each point in spacetime. A **(real) vector space** is a collection of objects (vectors) that can be added together and multiplied by real numbers in a linear way. Thus, for any two vectors V and W and real numbers a and b, we have

$$(a+b)(V+W) = aV + bV + aW + bW. \tag{1.36}$$

Every vector space has an origin, that is, a zero vector that functions as an identity element under vector addition. In many vector spaces there are additional operations such as taking an inner (dot) product, but this is extra structure over and above the elementary concept of a vector space.

A vector is a perfectly well-defined geometric object, as is a **vector field**, defined as a set of vectors with exactly one at each point in spacetime. [The set of all the tangent spaces of an n-dimensional manifold M can be assembled into a $2n$-dimensional manifold called the **tangent bundle**, $T(M)$. It is a specific example of a "fiber bundle," which is endowed with some extra mathematical structure; we won't need the details for our present purposes.] Nevertheless it is often useful to decompose vectors into components with respect to some set of basis vectors. A **basis** is any set of vectors which both spans the vector space (any vector is a linear combination of basis vectors) and is linearly independent (no vector in the basis

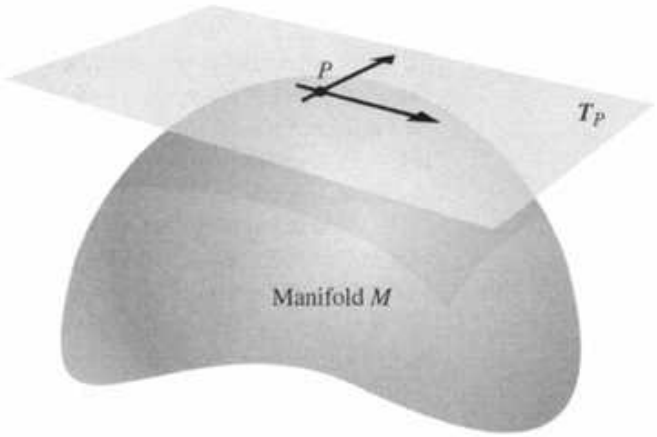


FIGURE 1.8 A suggestive drawing of the tangent space T_p, the space of all vectors at the point p.

is a linear combination of other basis vectors). For any given vector space, there will be an infinite number of possible bases we could choose, but each basis will consist of the same number of vectors, known as the **dimension** of the space. (For a tangent space associated with a point in Minkowski space, the dimension is, of course, four.)

Let us imagine that at each tangent space we set up a basis of four vectors $\hat{e}_{(\mu)}$, with $\mu \in \{0, 1, 2, 3\}$ as usual. In fact let us say that each basis is "adapted to the coordinates x^μ"—that is, the basis vector $\hat{e}_{(1)}$ is what we would normally think of pointing along the x-axis. It is by no means necessary that we choose a basis adapted to any coordinate system at all, although it is often convenient. (As before, we really could be more precise here, but later on we will repeat the discussion at an excruciating level of precision, so some sloppiness now is forgivable.) Then any abstract vector A can be written as a linear combination of basis vectors:

$$A = A^\mu \hat{e}_{(\mu)}. \tag{1.37}$$

The coefficients A^μ are the **components** of the vector A. More often than not we will forget the basis entirely and refer somewhat loosely to "the vector A^μ," but keep in mind that this is shorthand. The real vector is an abstract geometrical entity, while the components are just the coefficients of the basis vectors in some convenient basis. (Since we will usually suppress the explicit basis vectors, the indices usually will label components of vectors and tensors. This is why there are parentheses around the indices on the basis vectors, to remind us that this is a collection of vectors, not components of a single vector.)

A standard example of a vector in spacetime is the tangent vector to a curve. A parameterized curve or path through spacetime is specified by the coordinates as a function of the parameter, for example, $x^\mu(\lambda)$. The tangent vector $V(\lambda)$ has components

$$V^\mu = \frac{dx^\mu}{d\lambda}. \tag{1.38}$$

The entire vector is $V = V^\mu \hat{e}_{(\mu)}$. Under a Lorentz transformation the coordinates x^μ change according to (1.25), while the parameterization λ is unaltered; we can therefore deduce that the components of the tangent vector must change as

$$\boxed{V^\mu \rightarrow V^{\mu'} = \Lambda^{\mu'}{}_\nu V^\nu.} \tag{1.39}$$

However, the vector V itself (as opposed to its components in some coordinate system) is invariant under Lorentz transformations. We can use this fact to derive the transformation properties of the basis vectors. Let us refer to the set of basis vectors in the transformed coordinate system as $\hat{e}_{(\nu')}$. Since the vector is invariant, we have

$$V = V^\mu \hat{e}_{(\mu)} = V^{\nu'} \hat{e}_{(\nu')} = \Lambda^{\nu'}{}_\mu V^\mu \hat{e}_{(\nu')}. \tag{1.40}$$

But this relation must hold no matter what the numerical values of the components V^μ are. We can therefore say

$$\hat{e}_{(\mu)} = \Lambda^{\nu'}{}_{\mu}\hat{e}_{(\nu')}. \tag{1.41}$$

To get the new basis $\hat{e}_{(\nu')}$ in terms of the old one $\hat{e}_{(\mu)}$, we should multiply by the inverse of the Lorentz transformation $\Lambda^{\nu'}{}_{\mu}$. But the inverse of a Lorentz transformation from the unprimed to the primed coordinates is also a Lorentz transformation, this time from the primed to the unprimed systems. We will therefore introduce a somewhat subtle notation, by using the same symbol for both matrices, just with primed and unprimed indices switched. That is, the Lorentz transformation specified by $\Lambda^{\mu'}{}_{\nu}$ has an inverse transformation written as $\Lambda^{\rho}{}_{\sigma'}$. Operationally this implies

$$\Lambda^{\mu}{}_{\nu'}\Lambda^{\nu'}{}_{\rho} = \delta^{\mu}_{\rho}, \quad \Lambda^{\sigma'}{}_{\lambda}\Lambda^{\lambda}{}_{\tau'} = \delta^{\sigma'}_{\tau'}. \tag{1.42}$$

From (1.41) we then obtain the transformation rule for basis vectors:

$$\hat{e}_{(\nu')} = \Lambda^{\mu}{}_{\nu'}\hat{e}_{(\mu)}. \tag{1.43}$$

Therefore the set of basis vectors transforms via the inverse Lorentz transformation of the coordinates or vector components.

Let's pause a moment to take all this in. We introduced coordinates labeled by upper indices, which transformed in a certain way under Lorentz transformations. We then considered vector components that also were written with upper indices, which made sense since they transformed in the same way as the coordinate functions. (In a fixed coordinate system, each of the four coordinates x^μ can be thought of as a function on spacetime, as can each of the four components of a vector field.) The basis vectors associated with the coordinate system transformed via the inverse matrix, and were labeled by a lower index. This notation ensured that the invariant object constructed by summing over the components and basis vectors was left unchanged by the transformation, just as we would wish. It's probably not giving too much away to say that this will continue to be the case for tensors, which may have multiple indices.

1.5 ■ DUAL VECTORS (ONE-FORMS)

Once we have set up a vector space, we can define another associated vector space (of equal dimension) known as the **dual vector space**. The dual space is usually denoted by an asterisk, so that the dual space to the tangent space T_p, called the **cotangent space**, is denoted T_p^*. The dual space is the space of all linear maps from the original vector space to the real numbers; in math lingo, if $\omega \in T_p^*$ is a dual vector, then it acts as a map such that

$$\omega(aV + bW) = a\omega(V) + b\omega(W) \in \mathbf{R}, \tag{1.44}$$

where V, W are vectors and a, b are real numbers. The nice thing about these maps is that they form a vector space themselves; thus, if ω and η are dual vectors, we have

$$(a\omega + b\eta)(V) = a\omega(V) + b\eta(V). \tag{1.45}$$

To make this construction somewhat more concrete, we can introduce a set of basis dual vectors $\hat{\theta}^{(\nu)}$ by demanding

$$\hat{\theta}^{(\nu)}(\hat{e}_{(\mu)}) = \delta^{\nu}_{\mu}. \tag{1.46}$$

Then every dual vector can be written in terms of its components, which we label with lower indices:

$$\omega = \omega_{\mu}\hat{\theta}^{(\mu)}. \tag{1.47}$$

Usually, we will simply write ω_{μ}, in perfect analogy with vectors, to stand for the entire dual vector. In fact, you will sometimes see elements of T_p (what we have called vectors) referred to as **contravariant vectors**, and elements of T^*_p (what we have called dual vectors) referred to as **covariant vectors**, although in this day and age these terms sound a little dated. If you just refer to ordinary vectors as vectors with upper indices and dual vectors as vectors with lower indices, nobody should be offended. Another name for dual vectors is **one-forms**, a somewhat mysterious designation that will become clearer in Chapter 2.

The component notation leads to a simple way of writing the action of a dual vector on a vector:

$$\begin{aligned}\omega(V) &= \omega_{\mu}\hat{\theta}^{(\mu)}(V^{\nu}\hat{e}_{(\nu)}) \\ &= \omega_{\mu}V^{\nu}\hat{\theta}^{(\mu)}(\hat{e}_{(\nu)}) \\ &= \omega_{\mu}V^{\nu}\delta^{\mu}_{\nu} \\ &= \omega_{\mu}V^{\mu} \in \mathbf{R}.\end{aligned} \tag{1.48}$$

This is why it is rarely necessary to write the basis vectors and dual vectors explicitly; the components do all of the work. The form of (1.48) also suggests that we can think of vectors as linear maps on dual vectors, by defining

$$V(\omega) \equiv \omega(V) = \omega_{\mu}V^{\mu}. \tag{1.49}$$

Therefore, the dual space to the dual vector space is the original vector space itself.

Of course in spacetime we will be interested not in a single vector space, but in fields of vectors and dual vectors. [The set of all cotangent spaces over M can be combined into the **cotangent bundle**, $T^*(M)$.] In that case the action of a dual vector field on a vector field is not a single number, but a **scalar** (function) on spacetime. A scalar is a quantity without indices, which is unchanged under

Lorentz transformations; it is a coordinate-independent map from spacetime to the real numbers.

We can use the same arguments that we earlier used for vectors (that geometrical objects are independent of coordinates, even if their components are not) to derive the transformation properties of dual vectors. The answers are, for the components,

$$\boxed{\omega_{\mu'} = \Lambda^{\nu}{}_{\mu'}\omega_{\nu},} \tag{1.50}$$

and for basis dual vectors,

$$\hat{\theta}^{(\rho')} = \Lambda^{\rho'}{}_{\sigma}\hat{\theta}^{(\sigma)}. \tag{1.51}$$

This is just what we would expect from index placement; the components of a dual vector transform under the inverse transformation of those of a vector. Note that this ensures that the scalar (1.48) is invariant under Lorentz transformations, just as it should be.

In spacetime the simplest example of a dual vector is the **gradient** of a scalar function, the set of partial derivatives with respect to the spacetime coordinates, which we denote by a lowercase d:

$$\mathrm{d}\phi = \frac{\partial\phi}{\partial x^{\mu}}\hat{\theta}^{(\mu)}. \tag{1.52}$$

The conventional chain rule used to transform partial derivatives amounts in this case to the transformation rule of components of dual vectors:

$$\begin{aligned}\frac{\partial\phi}{\partial x^{\mu'}} &= \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial\phi}{\partial x^{\mu}} \\ &= \Lambda^{\mu}{}_{\mu'}\frac{\partial\phi}{\partial x^{\mu}},\end{aligned} \tag{1.53}$$

where we have used (1.25) to relate the Lorentz transformation to the coordinates. The fact that the gradient is a dual vector leads to the following shorthand notations for partial derivatives:

$$\frac{\partial\phi}{\partial x^{\mu}} = \partial_{\mu}\phi = \phi_{,\mu}. \tag{1.54}$$

So, x^{μ} has an upper index, but when it is in the denominator of a derivative it implies a lower index on the resulting object. In this book we will generally use ∂_{μ} rather than the comma notation. Note that the gradient does in fact act in a natural way on the example we gave above of a vector, the tangent vector to a curve. The result is an ordinary derivative of the function along the curve:

$$\partial_{\mu}\phi\frac{\partial x^{\mu}}{\partial\lambda} = \frac{d\phi}{d\lambda}. \tag{1.55}$$

1.6 ■ TENSORS

A straightforward generalization of vectors and dual vectors is the notion of a **tensor**. Just as a dual vector is a linear map from vectors to $\mathbf{R}$, a tensor T of type (or rank) (k, l) is a multilinear map from a collection of dual vectors and vectors to $\mathbf{R}$:

$$T : \underbrace{T_p^* \times \cdots \times T_p^*}_{(k \text{ times})} \times \underbrace{T_p \times \cdots \times T_p}_{(l \text{ times})} \to \mathbf{R}. \tag{1.56}$$

Here, "$\times$" denotes the Cartesian product, so that for example $T_p \times T_p$ is the space of ordered pairs of vectors. Multilinearity means that the tensor acts linearly in each of its arguments; for instance, for a tensor of type (1, 1), we have

$$\begin{aligned} T(a\omega + b\eta, cV + dW) &= acT(\omega, V) \\ &\quad + adT(\omega, W) + bcT(\eta, V) + bdT(\eta, W). \end{aligned} \tag{1.57}$$

From this point of view, a scalar is a type $(0, 0)$ tensor, a vector is a type $(1, 0)$ tensor, and a dual vector is a type $(0, 1)$ tensor.

The space of all tensors of a fixed type (k, l) forms a vector space; they can be added together and multiplied by real numbers. To construct a basis for this space, we need to define a new operation known as the **tensor product**, denoted by $\otimes$. If T is a (k, l) tensor and S is an (m, n) tensor, we define a $(k + m, l + n)$ tensor $T \otimes S$ by

$$\begin{aligned} &T \otimes S(\omega^{(1)}, \ldots, \omega^{(k)}, \ldots, \omega^{(k+m)}, V^{(1)}, \ldots, V^{(l)}, \ldots, V^{(l+n)}) \\ &\quad = T(\omega^{(1)}, \ldots, \omega^{(k)}, V^{(1)}, \ldots, V^{(l)}) \\ &\qquad \times S(\omega^{(k+1)}, \ldots, \omega^{(k+m)}, V^{(l+1)}, \ldots, V^{(l+n)}). \end{aligned} \tag{1.58}$$

Note that the $\omega^{(i)}$ and $V^{(i)}$ are distinct dual vectors and vectors, not components thereof. In other words, first act T on the appropriate set of dual vectors and vectors, and then act S on the remainder, and then multiply the answers. Note that, in general, tensor products do not commute: $T \otimes S \neq S \otimes T$.

It is now straightforward to construct a basis for the space of all (k, l) tensors, by taking tensor products of basis vectors and dual vectors; this basis will consist of all tensors of the form

$$\hat{e}_{(\mu_1)} \otimes \cdots \otimes \hat{e}_{(\mu_k)} \otimes \hat{\theta}^{(\nu_1)} \otimes \cdots \otimes \hat{\theta}^{(\nu_l)}. \tag{1.59}$$

In a four-dimensional spacetime there will be 4^{k+l} basis tensors in all. In component notation we then write our arbitrary tensor as

$$T = T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} \hat{e}_{(\mu_1)} \otimes \cdots \otimes \hat{e}_{(\mu_k)} \otimes \hat{\theta}^{(\nu_1)} \otimes \cdots \otimes \hat{\theta}^{(\nu_l)}. \tag{1.60}$$

Alternatively, we could define the components by acting the tensor on basis vectors and dual vectors:

$$T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l} = T(\hat{\theta}^{(\mu_1)}, \ldots, \hat{\theta}^{(\mu_k)}, \hat{e}_{(\nu_1)}, \ldots, \hat{e}_{(\nu_l)}). \tag{1.61}$$

You can check for yourself, using (1.46) and so forth, that these equations all hang together properly.

As with vectors, we will usually take the shortcut of denoting the tensor T by its components $T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l}$. The action of the tensors on a set of vectors and dual vectors follows the pattern established in (1.48):

$$T(\omega^{(1)}, \ldots, \omega^{(k)}, V^{(1)}, \ldots, V^{(l)}) = T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l}\omega^{(1)}_{\mu_1}\cdots\omega^{(k)}_{\mu_k}V^{(1)\nu_1}\cdots V^{(l)\nu_l}. \tag{1.62}$$

A (k, l) tensor thus has k upper indices and l lower indices. The order of the indices is obviously important, since the tensor need not act in the same way on its various arguments.

Finally, the transformation of tensor components under Lorentz transformations can be derived by applying what we already know about the transformation of basis vectors and dual vectors. The answer is just what you would expect from index placement,

$$T^{\mu'_1\cdots\mu'_k}{}_{\nu'_1\cdots\nu'_l} = \Lambda^{\mu'_1}{}_{\mu_1}\cdots\Lambda^{\mu'_k}{}_{\mu_k}\Lambda^{\nu_1}{}_{\nu'_1}\cdots\Lambda^{\nu_l}{}_{\nu'_l}T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l}. \tag{1.63}$$

Thus, each upper index gets transformed like a vector, and each lower index gets transformed like a dual vector.

Although we have defined tensors as linear maps from sets of vectors and dual vectors to $\mathbf{R}$, there is nothing that forces us to act on a full collection of arguments. Thus, a (1, 1) tensor also acts as a map from vectors to vectors:

$$T^{\mu}{}_{\nu} : V^{\nu} \rightarrow T^{\mu}{}_{\nu}V^{\nu}. \tag{1.64}$$

You can check for yourself that $T^{\mu}{}_{\nu}V^{\nu}$ is a vector (that is, obeys the vector transformation law). Similarly, we can act one tensor on (all or part of) another tensor to obtain a third tensor. For example,

$$U^{\mu}{}_{\nu} = T^{\mu\rho}{}_{\sigma}S^{\sigma}{}_{\rho\nu} \tag{1.65}$$

is a perfectly good (1, 1) tensor.

You may be concerned that this introduction to tensors has been somewhat too brief, given the esoteric nature of the material. In fact, the notion of tensors does not require a great deal of effort to master; it's just a matter of keeping the indices straight, and the rules for manipulating them are very natural. Indeed, a number of books like to *define* tensors as collections of numbers transforming according to (1.63). While this is operationally useful, it tends to obscure the deeper meaning of tensors as geometrical entities with a life independent of any chosen coordinate

system. There is, however, one subtlety that we have glossed over. The notions of dual vectors and tensors and bases and linear maps belong to the realm of linear algebra, and are appropriate whenever we have an abstract vector space at hand. In the case of interest to us we have not just a vector space, but a vector space at each point in spacetime. More often than not we are interested in tensor fields, which can be thought of as tensor-valued functions on spacetime. Fortunately, none of the manipulations we defined above really care whether we are dealing with a single vector space or a collection of vector spaces, one for each event. We will be able to get away with simply calling things functions of x^μ when appropriate. However, you should keep straight the logical independence of the notions we have introduced and their specific application to spacetime and relativity.

In spacetime, we have already seen some examples of tensors without calling them that. The most familiar example of a $(0, 2)$ tensor is the metric, $\eta_{\mu\nu}$. The action of the metric on two vectors is so useful that it gets its own name, the **inner product** (or scalar product, or dot product):

$$\eta(V, W) = \eta_{\mu\nu} V^\mu W^\nu = V \cdot W. \tag{1.66}$$

Just as with the conventional Euclidean dot product, we will refer to two vectors whose inner product vanishes as **orthogonal**. Since the inner product is a scalar, it is left invariant under Lorentz transformations; therefore, the basis vectors of any Cartesian inertial frame, which are chosen to be orthogonal by definition, are still orthogonal after a Lorentz transformation (despite the "scissoring together" we noticed earlier). The **norm** of a vector is defined to be inner product of the vector with itself; unlike in Euclidean space, this number is not positive definite:

$$\text{if } \eta_{\mu\nu} V^\mu V^\nu \text{ is } \begin{cases} < 0, & V^\mu \text{ is timelike} \\ = 0, & V^\mu \text{ is lightlike or null} \\ > 0, & V^\mu \text{ is spacelike.} \end{cases}$$

(A vector can have zero norm without being the zero vector.) You will notice that the terminology is the same as that which we used earlier to classify the relationship between two points in spacetime; it's no accident, of course, and we will go into more detail later.

Another tensor is the Kronecker delta δ^μ_ρ, of type $(1, 1)$. Thought of as a map from vectors to vectors (or one-forms to one-forms), the Kronecker delta is simply the identity map. We follow the example of many other references in placing the upper and lower indices in the same column for this unique tensor; purists might write $\delta^\mu{}_\rho$ or $\delta_\rho{}^\mu$, but these would be numerically identical, and we shouldn't get in trouble being careless in this one instance.

Related to the Kronecker delta and the metric is the **inverse metric** $\eta^{\mu\nu}$, a type $(2, 0)$ tensor defined (unsurprisingly) as the "inverse" of the metric:

$$\eta^{\mu\nu}\eta_{\nu\rho} = \eta_{\rho\nu}\eta^{\nu\mu} = \delta^\mu_\rho. \tag{1.67}$$

(It's the inverse metric since, when multiplied by the metric, it yields the identity map.) In fact, as you can check, the inverse metric has exactly the same compo-

nents as the metric itself. This is only true in flat space in Cartesian coordinates, and will fail to hold in more general situations. There is also the **Levi–Civita symbol**, a (0, 4) tensor:

$$\tilde{\epsilon}_{\mu\nu\rho\sigma} = \begin{cases} +1 & \text{if } \mu\nu\rho\sigma \text{ is an even permutation of } 0123 \\ -1 & \text{if } \mu\nu\rho\sigma \text{ is an odd permutation of } 0123 \\ 0 & \text{otherwise.} \end{cases} \tag{1.68}$$

Here, a "permutation of 0123" is an ordering of the numbers 0, 1, 2, 3, which can be obtained by starting with 0123 and exchanging two of the digits; an even permutation is obtained by an even number of such exchanges, and an odd permutation is obtained by an odd number. Thus, for example, $\tilde{\epsilon}_{0321} = -1$. (The tilde on $\tilde{\epsilon}_{\mu\nu\rho\sigma}$, and referring to it as a symbol rather than simply a tensor, derive from the fact that this object is actually not a tensor in more general geometries or coordinates; instead, it is something called a "tensor density." It is straightforward enough to define a related object that is a tensor, which we will denote by $\epsilon_{\mu\nu\rho\sigma}$ and call the "Levi–Civita tensor." See Chapter 2 for a discussion.)

A remarkable property of the above tensors—the metric, the inverse metric, the Kronecker delta, and the Levi–Civita symbol—is that, even though they all transform according to the tensor transformation law (1.63), their components remain unchanged in *any* inertial coordinate system in flat spacetime. In some sense this makes them nongeneric examples of tensors, since most tensors do not have this property. In fact, these are the *only* tensors with this property, although we won't prove it. The Kronecker delta is even more unusual, in that it has exactly the same components in any coordinate system in any spacetime. This makes sense from the definition of a tensor as a linear map; the Kronecker tensor can be thought of as the identity map from vectors to vectors (or from dual vectors to dual vectors), which clearly must have the same components regardless of coordinate system. Meanwhile, the metric and its inverse characterize the structure of spacetime, while the Levi–Civita symbol is secretly not a true tensor at all. We shall therefore have to treat these objects more carefully when we drop our assumption of flat spacetime.

A more typical example of a tensor is the **electromagnetic field strength tensor**. We all know that the electromagnetic fields are made up of the electric field vector E_i and the magnetic field vector B_i. (Remember that we use Latin indices for spacelike components 1, 2, 3.) Actually these are only "vectors" under rotations in space, not under the full Lorentz group. In fact they are components of a (0, 2) tensor $F_{\mu\nu}$, defined by

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} = -F_{\nu\mu}. \tag{1.69}$$

From this point of view it is easy to transform the electromagnetic fields in one reference frame to those in another, by application of (1.63). The unifying power of the tensor formalism is evident: rather than a collection of two vectors whose

relationship and transformation properties are rather mysterious, we have a single tensor field to describe all of electromagnetism. (On the other hand, don't get carried away; sometimes it's more convenient to work in a single coordinate system using the electric and magnetic field vectors.)

1.7 ■ MANIPULATING TENSORS

With these examples in hand we can now be a little more systematic about some properties of tensors. First consider the operation of **contraction**, which turns a (k, l) tensor into a $(k-1, l-1)$ tensor. Contraction proceeds by summing over one upper and one lower index:

$$S^{\mu\rho}{}_{\sigma} = T^{\mu\nu\rho}{}_{\sigma\nu}. \tag{1.70}$$

You can check that the result is a well-defined tensor. It is only permissible to contract an upper index with a lower index (as opposed to two indices of the same type); otherwise the result would *not* be a well-defined tensor. (By well-defined tensor we mean either "transforming according to the tensor transformation law," or "defining a unique multilinear map from a set of vectors and dual vectors to the real numbers"; take your pick.) Note also that the order of the indices matters, so that you can get different tensors by contracting in different ways; thus,

$$T^{\mu\nu\rho}{}_{\sigma\nu} \neq T^{\mu\rho\nu}{}_{\sigma\nu} \tag{1.71}$$

in general.

The metric and inverse metric can be used to **raise and lower indices** on tensors. That is, given a tensor $T^{\alpha\beta}{}_{\gamma\delta}$, we can use the metric to define new tensors, which we choose to denote by the same letter T:

$$\begin{aligned} T^{\alpha\beta\mu}{}_{\delta} &= \eta^{\mu\gamma} T^{\alpha\beta}{}_{\gamma\delta}, \\ T_{\mu}{}^{\beta}{}_{\gamma\delta} &= \eta_{\mu\alpha} T^{\alpha\beta}{}_{\gamma\delta}, \\ T_{\mu\nu}{}^{\rho\sigma} &= \eta_{\mu\alpha}\eta_{\nu\beta}\eta^{\rho\gamma}\eta^{\sigma\delta} T^{\alpha\beta}{}_{\gamma\delta}, \end{aligned} \tag{1.72}$$

and so forth. Notice that raising and lowering does not change the position of an index relative to other indices, and also that free indices (which are *not* summed over) must be the same on both sides of an equation, while dummy indices (which *are* summed over) only appear on one side. As an example, we can turn vectors and dual vectors into each other by raising and lowering indices:

$$\begin{aligned} V_{\mu} &= \eta_{\mu\nu} V^{\nu} \\ \omega^{\mu} &= \eta^{\mu\nu} \omega_{\nu}. \end{aligned} \tag{1.73}$$

Because the metric and inverse metric are truly inverses of each other, we are free to raise and lower simultaneously a pair of indices being contracted over:

$$A^{\lambda} B_{\lambda} = \eta^{\lambda\rho} A_{\rho} \eta_{\lambda\sigma} B^{\sigma} = \delta^{\rho}_{\sigma} A_{\rho} B^{\sigma} = A_{\sigma} B^{\sigma}. \tag{1.74}$$

The ability to raise and lower indices with a metric explains why the gradient in three-dimensional flat Euclidean space is usually thought of as an ordinary vector, even though we have seen that it arises as a dual vector; in Euclidean space (where the metric is diagonal with all entries $+1$) a dual vector is turned into a vector with precisely the same components when we raise its index. You may then wonder why we have belabored the distinction at all. One simple reason, of course, is that in a Lorentzian spacetime the components are not equal:

$$\omega^\mu = (-\omega_0, \omega_1, \omega_2, \omega_3). \tag{1.75}$$

In a curved spacetime, where the form of the metric is generally more complicated, the difference is rather more dramatic. But there is a deeper reason, namely that tensors generally have a "natural" definition independent of the metric. Even though we will always have a metric available, it is helpful to be aware of the logical status of each mathematical object we introduce. The gradient, with its action on vectors, is perfectly well-defined regardless of any metric, whereas the "gradient with upper indices" is not. (As an example, we will eventually want to take variations of functionals with respect to the metric, and will therefore have to know exactly how the functional depends on the metric, something that is easily obscured by the index notation.)

Continuing our compilation of tensor jargon, we refer to a tensor as **symmetric** in any of its indices if it is unchanged under exchange of those indices. Thus, if

$$S_{\mu\nu\rho} = S_{\nu\mu\rho}, \tag{1.76}$$

we say that $S_{\mu\nu\rho}$ is symmetric in its first two indices, while if

$$S_{\mu\nu\rho} = S_{\mu\rho\nu} = S_{\rho\mu\nu} = S_{\nu\mu\rho} = S_{\nu\rho\mu} = S_{\rho\nu\mu}, \tag{1.77}$$

we say that $S_{\mu\nu\rho}$ is symmetric in all three of its indices. Similarly, a tensor is **antisymmetric** (or skew-symmetric) in any of its indices if it changes sign when those indices are exchanged; thus,

$$A_{\mu\nu\rho} = -A_{\rho\nu\mu} \tag{1.78}$$

means that $A_{\mu\nu\rho}$ is antisymmetric in its first and third indices (or just "antisymmetric in μ and ρ"). If a tensor is (anti-) symmetric in all of its indices, we refer to it as simply (anti-) symmetric (sometimes with the redundant modifier "completely"). As examples, the metric $\eta_{\mu\nu}$ and the inverse metric $\eta^{\mu\nu}$ are symmetric, while the Levi–Civita symbol $\tilde{\epsilon}_{\mu\nu\rho\sigma}$ and the electromagnetic field strength tensor $F_{\mu\nu}$ are antisymmetric. (Check for yourself that if you raise or lower a set of indices that are symmetric or antisymmetric, they remain that way.) Notice that it makes no sense to exchange upper and lower indices with each other, so don't succumb to the temptation to think of the Kronecker delta δ^α_β as symmetric. On the other hand, the fact that lowering an index on δ^α_β gives a symmetric tensor (in fact, the metric) means that the order of indices doesn't really matter, which is why we don't keep track of index placement for this one tensor.

Given any tensor, we can **symmetrize** (or antisymmetrize) any number of its upper or lower indices. To symmetrize, we take the sum of all permutations of the relevant indices and divide by the number of terms:

$$T_{(\mu_1\mu_2\cdots\mu_n)\rho}{}^{\sigma} = \frac{1}{n!}\left(T_{\mu_1\mu_2\cdots\mu_n\rho}{}^{\sigma} + \text{sum over permutations of indices } \mu_1\cdots\mu_n\right), \tag{1.79}$$

while antisymmetrization comes from the alternating sum:

$$T_{[\mu_1\mu_2\cdots\mu_n]\rho}{}^{\sigma} = \frac{1}{n!}\left(T_{\mu_1\mu_2\cdots\mu_n\rho}{}^{\sigma} + \text{alternating sum over permutations of indices } \mu_1\cdots\mu_n\right). \tag{1.80}$$

By "alternating sum" we mean that permutations that are the result of an odd number of exchanges are given a minus sign, thus:

$$T_{[\mu\nu\rho]\sigma} = \tfrac{1}{6}\left(T_{\mu\nu\rho\sigma} - T_{\mu\rho\nu\sigma} + T_{\rho\mu\nu\sigma} - T_{\nu\mu\rho\sigma} + T_{\nu\rho\mu\sigma} - T_{\rho\nu\mu\sigma}\right). \tag{1.81}$$

Notice that round/square brackets denote symmetrization/antisymmetrization. Furthermore, we may sometimes want to (anti-) symmetrize indices that are not next to each other, in which case we use vertical bars to denote indices not included in the sum:

$$T_{(\mu|\nu|\rho)} = \tfrac{1}{2}\left(T_{\mu\nu\rho} + T_{\rho\nu\mu}\right). \tag{1.82}$$

If we are contracting over a pair of upper indices that are symmetric on one tensor, only the symmetric part of the lower indices will contribute; thus,

$$X^{(\mu\nu)}Y_{\mu\nu} = X^{(\mu\nu)}Y_{(\mu\nu)}, \tag{1.83}$$

regardless of the symmetry properties of $Y_{\mu\nu}$. (Analogous statements hold for antisymmetric indices, or if it's the lower indices that are symmetric to start with.) For any *two* indices, we can decompose a tensor into symmetric and antisymmetric parts,

$$T_{\mu\nu\rho\sigma} = T_{(\mu\nu)\rho\sigma} + T_{[\mu\nu]\rho\sigma}, \tag{1.84}$$

but this will not in general hold for three or more indices,

$$T_{\mu\nu\rho\sigma} \neq T_{(\mu\nu\rho)\sigma} + T_{[\mu\nu\rho]\sigma}, \tag{1.85}$$

because there are parts with mixed symmetry that are not specified by either the symmetric or antisymmetric pieces. Finally, some people use a convention in which the factor of $1/n!$ is omitted. The one used here is a good one, since, for example, a symmetric tensor satisfies

$$S_{\mu_1 \cdots \mu_n} = S_{(\mu_1 \cdots \mu_n)}, \tag{1.86}$$

and likewise for antisymmetric tensors.

For a (1, 1) tensor $X^\mu{}_\nu$, the **trace** is a scalar, often denoted by leaving off the indices, which is simply the contraction:

$$X = X^\lambda{}_\lambda. \tag{1.87}$$

If we think of $X^\mu{}_\nu$ as a matrix, this is just the sum of the diagonal components, so it makes sense. However, we will also use trace in the context of a (0, 2) tensor $Y_{\mu\nu}$, in which case it means that we should first raise an index ($Y^\mu{}_\nu = g^{\mu\lambda} Y_{\lambda\nu}$) and then contract:

$$Y = Y^\lambda{}_\lambda = \eta^{\mu\nu} Y_{\mu\nu}. \tag{1.88}$$

(It must be this way, since we cannot sum over two lower indices.) Although this is the sum of the diagonal components of $Y^\mu{}_\nu$, it is certainly *not* the sum of the diagonal components of $Y_{\mu\nu}$; we had to raise an index, which in general will change the numerical value of the components. For example, you might guess that the trace of the metric is $-1 + 1 + 1 + 1 = 2$, but it's not:

$$\eta^{\mu\nu} \eta_{\mu\nu} = \delta^\mu_\mu = 4. \tag{1.89}$$

(In n dimensions, $\delta^\mu_\mu = n$.) There is no reason to denote this trace by g (or δ), since it will always be the same number, even after we make the transition to curved spaces where the metric components are more complicated. Note that antisymmetric (0, 2) tensors are always traceless.

We have been careful so far to distinguish clearly between things that are always true (on a manifold with arbitrary metric) and things that are only true in Minkowski space in inertial coordinates. One of the most important distinctions arises with **partial derivatives**. If we are working in flat spacetime with inertial coordinates, then the partial derivative of a (k, l) tensor is a $(k, l + 1)$ tensor; that is,

$$T_\alpha{}^\mu{}_\nu = \partial_\alpha R^\mu{}_\nu \tag{1.90}$$

transforms properly under Lorentz transformations. However, this will no longer be true in more general spacetimes, and we will have to define a covariant derivative to take the place of the partial derivative. Nevertheless, we can still use the fact that partial derivatives give us tensors in this special case, as long as we keep our wits about us. [The one exception to this warning is the partial derivative of a scalar, $\partial_\alpha \phi$, which is a perfectly good tensor (the gradient) in any spacetime.] Of course, if we fix a particular coordinate system, the partial derivative is a perfectly good operator, which we will use all the time; its failure is only that it doesn't transform in the same way as the tensors we will be using (or equivalently, that the map it defines is not coordinate-independent). One of the most

useful properties of partial derivatives is that they commute,

$$\partial_\mu \partial_\nu (\cdots) = \partial_\nu \partial_\mu (\cdots), \tag{1.91}$$

no matter what kind of object is being differentiated.

1.8 ■ MAXWELL'S EQUATIONS

We have now accumulated enough tensor know-how to illustrate some of these concepts using actual physics. Specifically, we will examine **Maxwell's equations** of electrodynamics. In 19th-century notation, these are

$$\begin{aligned} \nabla \times \mathbf{B} - \partial_t \mathbf{E} &= \mathbf{J} \\ \nabla \cdot \mathbf{E} &= \rho \\ \nabla \times \mathbf{E} + \partial_t \mathbf{B} &= 0 \\ \nabla \cdot \mathbf{B} &= 0. \end{aligned} \tag{1.92}$$

Here, $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic field 3-vectors, $\mathbf{J}$ is the current, ρ is the charge density, and $\nabla\times$ and $\nabla\cdot$ are the conventional curl and divergence. These equations are invariant under Lorentz transformations, of course; that's how the whole business got started. But they don't look obviously invariant; our tensor notation can fix that. Let's begin by writing these equations in component notation,

$$\begin{aligned} \tilde{\epsilon}^{ijk} \partial_j B_k - \partial_0 E^i &= J^i \\ \partial_i E^i &= J^0 \\ \tilde{\epsilon}^{ijk} \partial_j E_k + \partial_0 B^i &= 0 \\ \partial_i B^i &= 0. \end{aligned} \tag{1.93}$$

In these expressions, spatial indices have been raised and lowered with abandon, without any attempt to keep straight where the metric appears, because δ_{ij} is the metric on flat 3-space, with δ^{ij} its inverse (they are equal as matrices). We can therefore raise and lower indices at will, since the components don't change. Meanwhile, the three-dimensional Levi–Civita symbol $\tilde{\epsilon}^{ijk}$ is defined just as the four-dimensional one, although with one fewer index (normalized so that $\tilde{\epsilon}^{123} = \tilde{\epsilon}_{123} = 1$). We have replaced the charge density by J^0; this is legitimate because the density and current together form the **current 4-vector**, $J^\mu = (\rho, J^x, J^y, J^z)$.

From (1.93), and the definition (1.69) of the field strength tensor $F_{\mu\nu}$, it is easy to get a completely tensorial 20th-century version of Maxwell's equations. Begin by noting that we can express the field strength with upper indices as

$$F^{0i} = E^i$$
$$F^{ij} = \tilde{\epsilon}^{ijk} B_k. \tag{1.94}$$

To check this, note for example that $F^{01} = \eta^{00}\eta^{11} F_{01}$ and $F^{12} = \tilde{\epsilon}^{123} B_3$. Then the first two equations in (1.93) become

$$\partial_j F^{ij} - \partial_0 F^{0i} = J^i$$
$$\partial_i F^{0i} = J^0. \tag{1.95}$$

Using the antisymmetry of $F^{\mu\nu}$, we see that these may be combined into the single tensor equation

$$\partial_\mu F^{\nu\mu} = J^\nu. \tag{1.96}$$

A similar line of reasoning, which is left as an exercise, reveals that the third and fourth equations in (1.93) can be written

$$\partial_{[\mu} F_{\nu\lambda]} = 0. \tag{1.97}$$

It's simple to verify that the antisymmetry of $F_{\mu\nu}$ implies that (1.97) can be equivalently expressed as

$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0. \tag{1.98}$$

The four traditional Maxwell equations are thus replaced by two, vividly demonstrating the economy of tensor notation. More importantly, however, both sides of equations (1.96) and (1.97) manifestly transform as tensors; therefore, if they are true in one inertial frame, they must be true in any Lorentz-transformed frame. This is why tensors are so useful in relativity—we often want to express relationships without recourse to any reference frame, and the quantities on each side of an equation must transform in the same way under changes of coordinates. As a matter of jargon, we will sometimes refer to quantities written in terms of tensors as **covariant** (which has nothing to do with "covariant" as opposed to "contravariant"). Thus, we say that (1.96) and (1.97) together serve as the covariant form of Maxwell's equations, while (1.92) or (1.93) are noncovariant.

1.9 ■ ENERGY AND MOMENTUM

We've now gone over essentially everything there is to know about the care and feeding of tensors. In the next chapter we will look more carefully at the rigorous definitions of manifolds and tensors, but the basic mechanics have been pretty well covered. Before jumping to more abstract mathematics, let's review how physics works in Minkowski spacetime.

Start with the worldline of a single particle. This is specified by a map $\mathbf{R} \to M$, where M is the manifold representing spacetime; we usually think of the path as

a parameterized curve $x^\mu(\lambda)$. As mentioned earlier, the tangent vector to this path is $dx^\mu/d\lambda$ (note that it depends on the parameterization). An object of primary interest is the norm of the tangent vector, which serves to characterize the path; if the tangent vector is timelike/null/spacelike at some parameter value λ, we say that the path is timelike/null/spacelike at that point. This explains why the same words are used to classify vectors in the tangent space and intervals between two points—because a straight line connecting, say, two timelike separated points will itself be timelike at every point along the path.

Nevertheless, be aware of the sleight of hand being pulled here. The metric, as a (0, 2) tensor, is a machine that acts on two vectors (or two copies of the same vector) to produce a number. It is therefore very natural to classify tangent vectors according to the sign of their norm. But the interval between two points isn't something quite so natural; it depends on a specific choice of path (a "straight line") that connects the points, and this choice in turn depends on the fact that spacetime is flat (which allows a unique choice of straight line between the points).

Let's move from the consideration of paths in general to the paths of massive particles (which will always be timelike). Since the proper time is measured by a clock traveling on a timelike worldline, it is convenient to use τ as the parameter along the path. That is, we use (1.22) to compute $\tau(\lambda)$, which (if λ is a good parameter in the first place) we can invert to obtain $\lambda(\tau)$, after which we can think of the path as $x^\mu(\tau)$. The tangent vector in this parameterization is known as the **four-velocity**, U^μ:

$$\boxed{U^\mu = \frac{dx^\mu}{d\tau}.} \tag{1.99}$$

Since $d\tau^2 = -\eta_{\mu\nu}dx^\mu dx^\nu$, the four-velocity is automatically normalized:

$$\eta_{\mu\nu}U^\mu U^\nu = -1. \tag{1.100}$$

This absolute normalization is a reflection of the fact that the four-velocity is not a velocity through space, which can of course take on different magnitudes, but a "velocity through spacetime," through which one always travels at the same rate. The norm of the four-velocity will always be negative, since we are only defining it for timelike trajectories. You could define an analogous vector for spacelike paths as well; for null paths the proper time vanishes, so τ can't be used as a parameter, and you have to be more careful. In the rest frame of a particle, its four-velocity has components $U^\mu = (1, 0, 0, 0)$.

A related vector is the **momentum four-vector**, defined by

$$\boxed{p^\mu = mU^\mu,} \tag{1.101}$$

where m is the mass of the particle. The mass is a fixed quantity independent of inertial frame, what you may be used to thinking of as the "rest mass." It turns out to be much more convenient to take this as the mass once and for all, rather than thinking of mass as depending on velocity. The **energy** of a particle is simply $E = p^0$, the timelike component of its momentum vector. Since it's only one component of a four-vector, it is not invariant under Lorentz transformations; that's to be expected, however, since the energy of a particle at rest is not the same as that of the same particle in motion. In the particle's rest frame we have $p^0 = m$; recalling that we have set $c = 1$, we see that we have found the equation that made Einstein a celebrity, $E = mc^2$. (The field equation of general relativity is actually more fundamental than this one, but $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}$ doesn't elicit the visceral reaction that you get from $E = mc^2$.) In a moving frame we can find the components of p^μ by performing a Lorentz transformation; for a particle moving with three-velocity $v = dx/dt$ along the x axis we have

$$p^\mu = (\gamma m, v\gamma m, 0, 0), \tag{1.102}$$

where $\gamma = 1/\sqrt{1 - v^2}$. For small v, this gives $p^0 = m + \frac{1}{2}mv^2$ (what we usually think of as rest energy plus kinetic energy) and $p^1 = mv$ (what we usually think of as Newtonian momentum). Outside this approximation, we can simply write

$$p_\mu p^\mu = -m^2, \tag{1.103}$$

or

$$E = \sqrt{m^2 + \mathbf{p}^2}, \tag{1.104}$$

where $\mathbf{p}^2 = \delta_{ij} p^i p^j$.

The centerpiece of pre-relativity physics is Newton's Second Law, or $\mathbf{f} = m\mathbf{a} = d\mathbf{p}/dt$. An analogous equation should hold in SR, and the requirement that it be tensorial leads us directly to introduce a force four-vector f^μ satisfying

$$f^\mu = m\frac{d^2}{d\tau^2}x^\mu(\tau) = \frac{d}{d\tau}p^\mu(\tau). \tag{1.105}$$

The simplest example of a force in Newtonian physics is the force due to gravity. In relativity, however, gravity is not described by a force, but rather by the curvature of spacetime itself. Instead, let us consider electromagnetism. The three-dimensional Lorentz force is given by $\mathbf{f} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$, where q is the charge on the particle. We would like a tensorial generalization of this equation. There turns out to be a unique answer:

$$f^\mu = -qU^\lambda F_\lambda{}^\mu. \tag{1.106}$$

You can check for yourself that this reduces to the Newtonian version in the limit of small velocities. Notice how the requirement that the equation be tensorial,

which is one way of guaranteeing Lorentz invariance, severely restricts the possible expressions we can get. This is an example of a very general phenomenon, in which a small number of an apparently endless variety of possible physical laws are picked out by the demands of symmetry.

Although p^μ provides a complete description of the energy and momentum of an individual particle, we often need to deal with extended systems comprised of huge numbers of particles. Rather than specify the individual momentum vectors of each particle, we instead describe the system as a **fluid**—a continuum characterized by macroscopic quantities such as density, pressure, entropy, viscosity, and so on. Although such a fluid may be composed of many individual particles with different four-velocities, the fluid itself has an overall four-velocity field. Just think of everyday fluids like air or water, where it makes sense to define a velocity for each individual fluid element even though nearby molecules may have appreciable relative velocities.

A single momentum four-vector field is insufficient to describe the energy and momentum of a fluid; we must go further and define the **energy-momentum tensor** (sometimes called the stress-energy tensor), $T^{\mu\nu}$. This symmetric (2, 0) tensor tells us all we need to know about the energy-like aspects of a system: energy density, pressure, stress, and so forth. A general definition of $T^{\mu\nu}$ is "the flux of four-momentum p^μ across a surface of constant x^ν." In fact, this definition is not going to be incredibly useful; in Chapter 4 we will define the energy-momentum tensor in terms of a functional derivative of the action with respect to the metric, which will be a more algorithmic procedure for finding an explicit expression for $T^{\mu\nu}$. But the definition here does afford some physical insight. Consider an infinitesimal element of the fluid in its rest frame, where there are no bulk motions. Then T^{00}, the "flux of p^0 (energy) in the x^0 (time) direction," is simply the rest-frame **energy density** ρ. Similarly, in this frame, $T^{0i} = T^{i0}$ is the momentum density. The spatial components T^{ij} are the momentum flux, or the *stress*; they represent the forces between neighboring infinitesimal elements of the fluid. Off-diagonal terms in T^{ij} represent shearing terms, such as those due to viscosity. A diagonal term such as T^{11} gives the x-component of the force being exerted (per unit area) by a fluid element in the x-direction; this is what we think of as the x-component of the **pressure**, p_x (don't confuse it with the momentum). The pressure has three components, given in the fluid rest frame (in inertial coordinates) by

$$p_i = T^{ii}. \tag{1.107}$$

There is no sum over i.

To make this more concrete, let's start with the simple example of **dust**. (Cosmologists tend to use "matter" as a synonym for dust.) Dust may be defined in flat spacetime as a collection of particles at rest with respect to each other. The four-velocity field $U^\mu(x)$ is clearly going to be the constant four-velocity of the individual particles. Indeed, its components will be the same at each point. Define the **number-flux four-vector** to be

$$N^\mu = nU^\mu, \tag{1.108}$$

where n is the number density of the particles as measured in their rest frame. (This doesn't sound coordinate-invariant, but it is; in any frame, the number density that would be measured if you were in the rest frame is a fixed quantity.) Then N^0 is the number density of particles as measured in any other frame, while N^i is the flux of particles in the x^i direction. Let's now imagine that each of the particles has the same mass m. Then in the rest frame the energy density of the dust is given by

$$\rho = mn. \tag{1.109}$$

By definition, the energy density completely specifies the dust. But ρ only measures the energy density in the rest frame; what about other frames? We notice that both n and m are 0-components of four-vectors in their rest frame; specifically, $N^\mu = (n, 0, 0, 0)$ and $p^\mu = (m, 0, 0, 0)$. Therefore ρ is the $\mu = 0, \nu = 0$ component of the tensor $p \otimes N$ as measured in its rest frame. We are therefore led to define the energy-momentum tensor for dust:

$$T^{\mu\nu}_{\text{dust}} = p^\mu N^\nu = mnU^\mu U^\nu = \rho U^\mu U^\nu, \tag{1.110}$$

where ρ is defined as the energy density in the rest frame. (Typically you don't just guess energy-momentum tensors by such a procedure, you derive them from equations of motion or an action principle.) Note that the pressure of the dust in any direction is zero; this should not be surprising, since pressure arises from the random motions of particles within the fluid, and we have defined dust to be free of such motions.

Dust is not sufficiently general to describe most of the interesting fluids that appear in general relativity; we only need a slight generalization, however, to arrive at the concept of a **perfect fluid**. A perfect fluid is one that can be completely specified by two quantities, the rest-frame energy density ρ, and an isotropic rest-frame pressure p. The single parameter p serves to specify the pressure in every direction. A consequence of isotropy is that $T^{\mu\nu}$ is diagonal in its rest frame—there is no net flux of any component of momentum in an orthogonal direction. Furthermore, the nonzero spacelike components must all be equal, $T^{11} = T^{22} = T^{33}$. The only two independent numbers are therefore the energy density $\rho = T^{00}$ and the pressure $p = T^{ii}$; we don't need a subscript on p, since the pressure is equal in every direction. The energy-momentum tensor of a perfect fluid therefore takes the following form in its rest frame:

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}. \tag{1.111}$$

(Remember that we are in flat spacetime; this will change when curvature is introduced.) We would like, of course, a formula that is good in any frame. For dust we had $T^{\mu\nu} = \rho U^\mu U^\nu$, so we might begin by guessing $(\rho + p)U^\mu U^\nu$, which

gives

$$\begin{pmatrix} \rho + p & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{1.112}$$

This is not a very clever guess, to be honest. But by subtracting this guess from our desired answer, we see that what we need to add is

$$\begin{pmatrix} -p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}. \tag{1.113}$$

Fortunately, this has an obvious covariant generalization, namely $p\eta^{\mu\nu}$. Thus, the general form of the energy-momentum tensor for a perfect fluid is

$$\boxed{T^{\mu\nu} = (\rho + p)U^\mu U^\nu + p\eta^{\mu\nu}.} \tag{1.114}$$

It may seem that the procedure used to arrive at this formula was somewhat arbitrary, but we can have complete confidence in the result. Given that (1.111) should be the form of $T^{\mu\nu}$ in the rest frame, and that (1.114) is a perfectly tensorial expression that reduces to (1.111) in the rest frame, we know that (1.114) must be the right expression in any frame.

The concept of a perfect fluid is general enough to describe a wide variety of physical forms of matter. To determine the evolution of such a fluid, we specify an equation of state relating the pressure to the energy density, $p = p(\rho)$. Dust is a special case for which $p = 0$, while an isotropic gas of photons has $p = \frac{1}{3}\rho$. A more exotic example is vacuum energy, for which the energy-momentum tensor is proportional to the metric, $T^{\mu\nu} = -\rho_{\text{vac}}\eta^{\mu\nu}$. By comparing to (1.114) we find that vacuum energy is a kind of perfect fluid for which $p_{\text{vac}} = -\rho_{\text{vac}}$. The notion of an energy density in vacuum is completely pointless in special relativity, since in nongravitational physics the absolute value of the energy doesn't matter, only the difference in energy between two states. In general relativity, however, all energy couples to gravity, so the possibility of a nonzero vacuum energy will become an important consideration, which we will discuss more fully in Chapter 4.

Besides being symmetric, $T^{\mu\nu}$ has the even more important property of being *conserved*. In this context, conservation is expressed as the vanishing of the "divergence":

$$\boxed{\partial_\mu T^{\mu\nu} = 0.} \tag{1.115}$$

This expression is a set of four equations, one for each value of ν. The equation with $\nu = 0$ corresponds to conservation of energy, while $\partial_\mu T^{\mu k} = 0$ expresses

conservation of the kth component of the momentum. Let's apply this equation to a perfect fluid, for which we have

$$\partial_\mu T^{\mu\nu} = \partial_\mu(\rho + p)U^\mu U^\nu + (\rho + p)(U^\nu \partial_\mu U^\mu + U^\mu \partial_\mu U^\nu) + \partial^\nu p. \quad (1.116)$$

To analyze what this equation means, it is helpful to consider separately what happens when we project it into pieces along and orthogonal to the four-velocity field U^μ. We first note that the normalization $U_\nu U^\nu = -1$ implies the useful identity

$$U_\nu \partial_\mu U^\nu = \tfrac{1}{2}\partial_\mu(U_\nu U^\nu) = 0. \quad (1.117)$$

To project (1.116) along the four-velocity, simply contract it into U_ν:

$$U_\nu \partial_\mu T^{\mu\nu} = -\partial_\mu(\rho U^\mu) - p\partial_\mu U^\mu. \quad (1.118)$$

Setting this to zero gives the relativistic equation of energy conservation for a perfect fluid. It will look more familiar in the nonrelativistic limit, in which

$$U^\mu = (1, v^i), \qquad |v^i| \ll 1, \qquad p \ll \rho. \quad (1.119)$$

The last condition makes sense, because pressure comes from the random motions of the individual particles, and in this limit these motions (as well as the bulk motion described by U^μ) are taken to be small. So in ordinary nonrelativistic language, (1.118) becomes

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0, \quad (1.120)$$

the continuity equation for the energy density. We next consider the part of (1.116) that is orthogonal to the four-velocity. To project a vector orthogonal to U^μ, we multiply it by the projection tensor

$$P^\sigma{}_\nu = \delta^\sigma_\nu + U^\sigma U_\nu. \quad (1.121)$$

To convince yourself this does the trick, check that if we have a vector $V^\mu_\parallel$, parallel to U^μ, and another vector $W^\mu_\perp$, perpendicular to U^μ, the projection tensor will annihilate the parallel vector and preserve the orthogonal one:

$$\begin{aligned} P^\sigma{}_\nu V^\nu_\parallel &= 0 \\ P^\sigma{}_\nu W^\nu_\perp &= W^\sigma_\perp. \end{aligned} \quad (1.122)$$

Applied to $\partial_\mu T^{\mu\nu}$, we obtain

$$P^\sigma{}_\nu \partial_\mu T^{\mu\nu} = (\rho + p)U^\mu \partial_\mu U^\sigma + \partial^\sigma p + U^\sigma U^\mu \partial_\mu p. \quad (1.123)$$

In the nonrelativistic limit given by (1.119), setting the spatial components of this expression equal to zero yields

$$\rho\,[\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v}] + \nabla p + \mathbf{v}(\partial_t p + \mathbf{v} \cdot \nabla p) = 0. \quad (1.124)$$

But notice that the last set of terms involve derivatives of p times the three-velocity $\mathbf{v}$, assumed to be small; these will therefore be negligible compared to the ∇p term, and can be neglected. We are left with

$$\rho\left[\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right] = -\nabla p, \tag{1.125}$$

which is the Euler equation familiar from fluid mechanics.

1.10 ■ CLASSICAL FIELD THEORY

When we make the transition from special relativity to general relativity, the metric $\eta_{\mu\nu}$ will be promoted to a dynamical tensor field, $g_{\mu\nu}(x)$. GR is thus a particular example of a classical field theory; we can build up some feeling for how such theories work by considering classical fields defined on flat spacetime. (We say classical field theory in contrast with quantum field theory, which is quite a different story; we will discuss it briefly in Chapter 9, but it is outside our main area of interest here.)

Let's begin with the familiar example of the classical mechanics of a single particle in one dimension with coordinate $q(t)$. We can derive the equations of motion for such a particle by using the "principle of least action": we search for critical points (as a function of the trajectory) of an **action** S, written as

$$S = \int dt\, L(q, \dot{q}), \tag{1.126}$$

where the function $L(q, \dot{q})$ is the **Lagrangian**. The Lagrangian in point-particle mechanics is typically of the form

$$L = K - V, \tag{1.127}$$

where K is the kinetic energy and V the potential energy. Following the calculus-of-variations procedure, which is described in any advanced textbook on classical mechanics, we show that critical points of the action [trajectories $q(t)$ for which S remains stationary under small variations] are those that satisfy the **Euler–Lagrange equations**,

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial(\dot{q})}\right) = 0. \tag{1.128}$$

For example, $L = \frac{1}{2}\dot{q}^2 - V(q)$ leads to

$$\ddot{q} = -\frac{dV}{dq}. \tag{1.129}$$

Field theory is a similar story, except that we replace the single coordinate $q(t)$ by a set of spacetime-dependent **fields**, $\Phi^i(x^\mu)$, and the action S becomes a *functional* of these fields. A functional is simply a function of an infinite number of

variables, such as the values of a field in some region of spacetime. Functionals are often expressed as integrals. Each Φ^i is a function on spacetime (at least in some coordinate system), and i is an index labeling our individual fields. For example, in electromagnetism (as we will see below) the fields are the four components of a one-form called the "vector potential," A_μ:

$$\Phi^i = \{A_0, A_1, A_2, A_3\}. \tag{1.130}$$

We're being very lowbrow here, in thinking of a one-form field as four different functions rather than a single tensor object. This point of view makes sense so long as we stick to a fixed coordinate system, and it will make our calculations more straightforward.

In field theory, the Lagrangian can be expressed as an integral over space of a **Lagrange density** $\mathcal{L}$, which is a function of the fields Φ^i and their spacetime derivatives $\partial_\mu \Phi^i$:

$$L = \int d^3x \, \mathcal{L}(\Phi^i, \partial_\mu \Phi^i). \tag{1.131}$$

So the action is

$$S = \int dt \, L = \int d^4x \, \mathcal{L}(\Phi^i, \partial_\mu \Phi^i). \tag{1.132}$$

The Lagrange density is a Lorentz scalar. We typically just say "Lagrangian" when we mean "Lagrange density." It will most often be convenient to define a field theory by specifying the Lagrange density, from which all of the equations of motion can be readily derived.

We will use "natural units," in which not only $c = 1$ but also $\hbar = k = 1$, where $\hbar = h/2\pi$, h is Planck's constant, and k is Boltzmann's constant. The objection might be raised that we shouldn't involve $\hbar$ in a purely classical discussion; but all we are doing here is choosing units, not determining physics. (The relevance of $\hbar$ would appear if we were to quantize our field theory and obtain particles, but we won't get that far right now.) In natural units we have

$$[\text{energy}] = [\text{mass}] = [(\text{length})^{-1}] = [(\text{time})^{-1}]. \tag{1.133}$$

We will most often use energy or mass as our fundamental unit. Since the action is an integral of L (with units of energy) over time, it is dimensionless:

$$[S] = [E][T] = M^0. \tag{1.134}$$

The volume element has units

$$[d^4x] = M^{-4}, \tag{1.135}$$

so to get a dimensionless action we require that the Lagrange density have units

$$[\mathcal{L}] = M^4. \tag{1.136}$$

The Euler–Lagrange equations come from requiring that the action be unchanged under small variations of the fields,

$$\Phi^i \to \Phi^i + \delta\Phi^i, \tag{1.137}$$

$$\partial_\mu \Phi^i \to \partial_\mu \Phi^i + \delta(\partial_\mu \Phi^i) = \partial_\mu \Phi^i + \partial_\mu(\delta\Phi^i). \tag{1.138}$$

The expression for the variation in $\partial_\mu \Phi^i$ is simply the derivative of the variation of Φ^i. Since $\delta\Phi^i$ is assumed to be small, we may Taylor-expand the Lagrangian under this variation:

$$\begin{aligned}\mathcal{L}(\Phi^i, \partial_\mu \Phi^i) &\to \mathcal{L}(\Phi^i + \delta\Phi^i, \partial_\mu \Phi^i + \partial_\mu \delta\Phi^i) \\ &= \mathcal{L}(\Phi^i, \partial_\mu \Phi^i) + \frac{\partial \mathcal{L}}{\partial \Phi^i}\delta\Phi^i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^i)}\partial_\mu(\delta\Phi^i).\end{aligned} \tag{1.139}$$

Correspondingly, the action goes to $S \to S + \delta S$, with

$$\delta S = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \Phi^i}\delta\Phi^i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^i)}\partial_\mu(\delta\Phi^i)\right]. \tag{1.140}$$

We would like to factor out $\delta\Phi^i$ from the integrand, by integrating the second term by parts:

$$\begin{aligned}\int d^4x \, \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^i)}\partial_\mu(\delta\Phi^i) = &- \int d^4x \, \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^i)}\right)\delta\Phi^i \\ &+ \int d^4x \, \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^i)}\delta\Phi^i\right).\end{aligned} \tag{1.141}$$

The final term is a total derivative—the integral of something of the form $\partial_\mu V^\mu$—that can be converted to a surface term by Stokes's theorem (the four-dimensional version, that is; see Appendix E for a discussion). Since we are considering variational problems, we can choose to consider variations that vanish at the boundary (along with their derivatives). It is therefore traditional in such contexts to integrate by parts with complete impunity, always ignoring the boundary contributions. (Sometimes this is not okay, as in instanton calculations in Yang–Mills theory.)

We are therefore left with

$$\delta S = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \Phi^i} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^i)}\right)\right]\delta\Phi^i. \tag{1.142}$$

The functional derivative $\delta S/\delta\Phi^i$ of a functional S with respect to a function Φ^i is defined to satisfy

$$\delta S = \int d^4x \frac{\delta S}{\delta \Phi^i}\delta\Phi^i, \tag{1.143}$$

when such an expression is valid. We can therefore express the notion that S is at a critical point by saying that the functional derivative vanishes. The final equations of motion for our field theory are thus:

$$\boxed{\frac{\delta S}{\delta \Phi^i} = \frac{\partial \mathcal{L}}{\partial \Phi^i} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^i)} \right) = 0.} \tag{1.144}$$

These are known as the Euler–Lagrange equations for a field theory in flat spacetime.

The simplest example of a field is a real scalar field:

$$\phi(x^\mu) : (\text{spacetime}) \to \mathbf{R}. \tag{1.145}$$

Slightly more complicated examples would include complex scalar fields, or maps from spacetime to any vector space or even any manifold (sometimes called "nonlinear sigma models"). Upon quantization, excitations of the field are observable as particles. Scalar fields give rise to spinless particles, while vector fields and other tensors give rise to higher-spin particles. If the field were complex instead of real, it would have two degrees of freedom rather than just one, which would be interpreted as a particle and a distinct antiparticle. Real fields are their own antiparticles. An example of a real scalar field would be the neutral π-meson.

So let's consider the classical mechanics of a single real scalar field. It will have an energy density that is a local function of spacetime, and includes various contributions:

$$\begin{array}{rc} \text{kinetic energy :} & \frac{1}{2}\dot{\phi}^2 \\ \text{gradient energy :} & \frac{1}{2}(\nabla\phi)^2 \\ \text{potential energy :} & V(\phi). \end{array} \tag{1.146}$$

Actually, although the potential is a Lorentz-invariant function, the kinetic and gradient energies are not by themselves Lorentz-invariant; but we can combine them into a manifestly Lorentz-invariant form:

$$-\tfrac{1}{2}\eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) = \tfrac{1}{2}\dot{\phi}^2 - \tfrac{1}{2}(\nabla\phi)^2. \tag{1.147}$$

[The combination $\eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi)$ is often abbreviated as $(\partial\phi)^2$.] So a reasonable choice of Lagrangian for our single real scalar field, analogous to $L = K - V$ in the point-particle case, would be

$$\mathcal{L} = -\tfrac{1}{2}\eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - V(\phi). \tag{1.148}$$

This generalizes "kinetic minus potential energy" to "kinetic minus gradient minus potential energy density." Note that since $[\mathcal{L}] = M^4$, we must have $[V] = M^4$. Also, since $[\partial_\mu] = [\partial/\partial x^\mu] = M^1$, we have

$$[\phi] = M^1. \tag{1.149}$$

For the Lagrangian (1.148) we have

$$\frac{\partial \mathcal{L}}{\partial \phi} = -\frac{dV}{d\phi}, \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = -\eta^{\mu\nu}\partial_\nu \phi. \tag{1.150}$$

The second of these equations is a little tricky, so let's go through it slowly. When differentiating the Lagrangian, the trick is to make sure that the index placement is "compatible" (so that if you have a lower index on the thing being differentiated with respect to, you should have only lower indices when the same kind of object appears in the thing being differentiated), and also that the indices are strictly different. The first of these is already satisfied in our example, since we are differentiating a function of $\partial_\mu \phi$ with respect to $\partial_\mu \phi$. Later on, we will need to be more careful. To fulfill the second, we simply relabel dummy indices:

$$\eta^{\mu\nu}(\partial_\mu \phi)(\partial_\nu \phi) = \eta^{\rho\sigma}(\partial_\rho \phi)(\partial_\sigma \phi). \tag{1.151}$$

Then we can use the general rule, for any object with one index such as V_μ, that

$$\frac{\partial V_\alpha}{\partial V_\beta} = \delta^\beta_\alpha \tag{1.152}$$

because each component of V_α is treated as a distinct variable. So we have

$$\begin{aligned}\frac{\partial}{\partial(\partial_\mu \phi)}\left[\eta^{\rho\sigma}(\partial_\rho \phi)(\partial_\sigma \phi)\right] &= \eta^{\rho\sigma}\left[\delta^\mu_\rho(\partial_\sigma \phi) + (\partial_\rho \phi)\delta^\mu_\sigma\right] \\ &= \eta^{\mu\sigma}(\partial_\sigma \phi) + \eta^{\rho\mu}(\partial_\rho \phi) = 2\eta^{\mu\nu}\partial_\nu \phi. \end{aligned} \tag{1.153}$$

This leads to the second expression in (1.150).

Putting (1.150) into (1.144) leads to the equation of motion

$$\Box \phi - \frac{dV}{d\phi} = 0, \tag{1.154}$$

where $\Box \equiv \eta^{\mu\nu}\partial_\mu \partial_\nu$ is known as the **d'Alembertian**. Note that our metric sign convention $(-+++)$ comes into this equation; with the alternative $(+---)$ convention the sign would have been switched. In flat spacetime (1.154) is equivalent to

$$\ddot{\phi} - \nabla^2 \phi + \frac{dV}{d\phi} = 0. \tag{1.155}$$

A popular choice for the potential V is that of a simple harmonic oscillator, $V(\phi) = \frac{1}{2}m^2\phi^2$. The parameter m is called the mass of the field, and you should notice that the units work out correctly. You may be wondering how a field can have mass. When we quantize the field we find that momentum eigenstates are collections of particles, each with mass m. At the classical level, we think of "mass" as simply a convenient characterization of the field dynamics. Then our

equation of motion is

$$\Box\phi - m^2\phi = 0, \tag{1.156}$$

the famous **Klein–Gordon equation**. This is a linear differential equation, so the sum of two solutions is a solution; a complete set of solutions (in the form of plane waves) is easy to find, as you can check for yourself.

A slightly more elaborate example of a field theory is provided by electromagnetism. We mentioned that the relevant field is the **vector potential** A_μ; the timelike component A_0 can be identified with the electrostatic potential Φ, and the spacelike components with the traditional vector potential $\mathbf{A}$ (in terms of which the magnetic field is given by $\mathbf{B} = \nabla \times \mathbf{A}$). The field strength tensor, with components given by (1.69), is related to the vector potential by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{1.157}$$

From this definition we see that the field strength tensor has the important property of **gauge invariance**: when we perform a **gauge transformation** on the vector potential,

$$A_\mu \to A_\mu + \partial_\mu \lambda(x), \tag{1.158}$$

the field strength tensor is left unchanged:

$$F_{\mu\nu} \to F_{\mu\nu} + \partial_\mu\partial_\nu\lambda - \partial_\nu\partial_\mu\lambda = F_{\mu\nu}. \tag{1.159}$$

The last equality follows from the fact that partial derivatives commute, $\partial_\mu\partial_\nu = \partial_\nu\partial_\mu$. Gauge invariance is a symmetry that is fundamental to our understanding of electromagnetism, and all observable quantities must be gauge-invariant. Thus, while the dynamical field of the theory (with respect to which we vary the action to derive equations of motion) is A_μ, physical quantities will generally be expressed in terms of $F_{\mu\nu}$.

We already know that the dynamical equations of electromagnetism are Maxwell's equations, (1.96) and (1.97). Given the definition of the field strength tensor in terms of the vector potential, (1.97) is actually automatic:

$$\partial_{[\mu}F_{\nu\sigma]} = \partial_{[\mu}\partial_\nu A_{\sigma]} - \partial_{[\mu}\partial_\sigma A_{\nu]} = 0, \tag{1.160}$$

again because partial derivatives commute. On the other hand, (1.96) is equivalent to Euler–Lagrange equations of the form

$$\frac{\partial\mathcal{L}}{\partial A_\nu} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) = 0, \tag{1.161}$$

if we presciently choose the Lagrangian to be

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + A_\mu J^\mu. \tag{1.162}$$

For this choice, the first term in the Euler–Lagrange equation is straightforward:

$$\frac{\partial \mathcal{L}}{\partial A_\nu} = \delta^\nu_\mu J^\mu = J^\nu. \tag{1.163}$$

The second term is trickier. First we write $F_{\mu\nu}F^{\mu\nu}$ as

$$F_{\mu\nu}F^{\mu\nu} = F_{\alpha\beta}F^{\alpha\beta} = \eta^{\alpha\rho}\eta^{\beta\sigma}F_{\alpha\beta}F_{\rho\sigma}. \tag{1.164}$$

We want to work with lower indices on $F_{\mu\nu}$, since we are differentiating with respect to $\partial_\mu A_\nu$, which has lower indices. Likewise we change the dummy indices on $F_{\mu\nu}F^{\mu\nu}$, since we want to have different indices on the thing being differentiated and the thing we are differentiating with respect to. Once you get familiar with this stuff it will become second nature and you won't need nearly so many steps. This lets us write

$$\frac{\partial(F_{\alpha\beta}F^{\alpha\beta})}{\partial(\partial_\mu A_\nu)} = \eta^{\alpha\rho}\eta^{\beta\sigma}\left[\left(\frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\nu)}\right)F_{\rho\sigma} + F_{\alpha\beta}\left(\frac{\partial F_{\rho\sigma}}{\partial(\partial_\mu A_\nu)}\right)\right]. \tag{1.165}$$

Then, since $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$, we have

$$\frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\nu)} = \delta^\mu_\alpha\delta^\nu_\beta - \delta^\mu_\beta\delta^\nu_\alpha. \tag{1.166}$$

Combining (1.166) with (1.165) yields

$$\begin{aligned}\frac{\partial(F_{\alpha\beta}F^{\alpha\beta})}{\partial(\partial_\mu A_\nu)} &= \eta^{\alpha\rho}\eta^{\beta\sigma}\left[(\delta^\mu_\alpha\delta^\nu_\beta - \delta^\mu_\beta\delta^\nu_\alpha)F_{\rho\sigma} + (\delta^\mu_\rho\delta^\nu_\sigma - \delta^\mu_\sigma\delta^\nu_\rho)F_{\alpha\beta}\right] \\ &= (\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\nu\rho}\eta^{\mu\sigma})F_{\rho\sigma} + (\eta^{\alpha\mu}\eta^{\beta\nu} - \eta^{\alpha\nu}\eta^{\beta\mu})F_{\alpha\beta} \\ &= F^{\mu\nu} - F^{\nu\mu} + F^{\mu\nu} - F^{\nu\mu} \\ &= 4F^{\mu\nu},\end{aligned} \tag{1.167}$$

so

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -F^{\mu\nu}. \tag{1.168}$$

Then sticking (1.163) and (1.168) into (1.161) yields precisely (1.96):

$$\partial_\mu F^{\nu\mu} = J^\nu. \tag{1.169}$$

Note that we switched the order of the indices on $F^{\mu\nu}$ in order to save ourselves from an unpleasant minus sign.

You may wonder what the purpose of introducing a Lagrangian formulation is, if we were able to invent the equations of motion before we ever knew the Lagrangian (as Maxwell did for his equations). There are a number of reasons,

starting with the basic simplicity of positing a single scalar function of spacetime, the Lagrange density, rather than a number of (perhaps tensor-valued) equations of motion. Another reason is the ease with which symmetries are implemented; demanding that the action be invariant under a symmetry ensures that the dynamics respects the symmetry as well. Finally, as we will see in Chapter 4, the action leads via a direct procedure (involving varying with respect to the metric itself) to a unique energy-momentum tensor. Applying this procedure to (1.148) leads straight to the energy-momentum tensor for a scalar field theory,

$$T^{\mu\nu}_{\text{scalar}} = \eta^{\mu\lambda}\eta^{\nu\sigma}\partial_\lambda\phi\partial_\sigma\phi - \eta^{\mu\nu}\left[\tfrac{1}{2}\eta^{\lambda\sigma}\partial_\lambda\phi\partial_\sigma\phi + V(\phi)\right]. \tag{1.170}$$

Similarly, from (1.162) we can derive the energy-momentum tensor for electromagnetism,

$$T^{\mu\nu}_{\text{EM}} = F^{\mu\lambda}F^{\nu}{}_{\lambda} - \tfrac{1}{4}\eta^{\mu\nu}F^{\lambda\sigma}F_{\lambda\sigma}. \tag{1.171}$$

Using the appropriate equations of motion, you can show that these energy-momentum tensors are conserved, $\partial_\mu T^{\mu\nu} = 0$ (and will be asked to do so in the Exercises).

The two examples we have considered—scalar field theory and electromagnetism—are paradigms for much of our current understanding of nature. The Standard Model of particle physics consists of three types of fields: gauge fields, Higgs fields, and fermions. The gauge fields describe the "forces" of nature, including the strong and weak nuclear forces in addition to electromagnetism. The gauge fields giving rise to the nuclear forces are described by one-form potentials, just as in electromagnetism; the difference is that they are matrix-valued rather than ordinary one-forms, and the symmetry groups corresponding to gauge transformations are therefore noncommutative (nonabelian) symmetries. The Higgs fields are scalar fields much as we have described, although they are also matrix-valued. The fermions include leptons (such as electrons and neutrinos) and quarks, and are not described by any of the tensor fields we have discussed here, but rather by a different kind of field called a **spinor**. We won't get around to discussing spinors in this book, but they play a crucial role in particle physics and their coupling to gravity is interesting and subtle. Upon quantization, these fields give rise to particles of different spins; gauge fields are spin-1, scalar fields are spin-0, and the Standard Model fermions are spin-$\frac{1}{2}$.

Before concluding this chapter, let's ask an embarassingly simple question: Why should we consider one classical field theory rather than some other one? More concretely, let's say that we have discovered some particle in nature, and we know what kind of field we want to use to describe it; how should we pick the Lagrangian for this field? For example, when we wrote down our scalar-field Lagrangian (1.148), why didn't we include a term of the form

$$\mathcal{L}' = \lambda\phi^2\eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi), \tag{1.172}$$

where λ is a coupling constant? Ultimately, of course, we work by trial and error and try to fit the data given to us by experiment. In classical field theory, there's not much more we could do; generally we would start with a simple Lagrangian, and perhaps make it more complicated if the first try failed to agree with the data. But quantum field theory actually provides some simple guidelines, and since we use classical field theory as an approximation to some underlying quantum theory, it makes sense to take advantage of these principles. To make a long story short, quantum field theory allows "virtual" processes at arbitrarily high energies to contribute to what we observe at low energies. Fortunately, the effect of these processes can be summarized in a low-energy **effective field theory**. In the effective theory, which is what we actually observe, the result of high-energy processes is simply to "renormalize" the coupling constants of our theory. Consider an arbitrary coupling constant, which we can express as a parameter μ (with dimensions of mass) raised to some power, $\lambda = \mu^q$ (unless λ is dimensionless, in which case the discussion becomes more subtle). Very roughly speaking, *the effect of high-energy processes will be to make μ very large*. Slightly more specifically, μ will be pushed up to a scale at which new physics kicks in, whatever that may be. Therefore, potential higher-order terms we might think of adding to a Lagrangian are suppressed, because they are multiplied by coupling constants that are very small. For (1.172), for example, we must have $\lambda = \mu^{-2}$, so λ will be tiny (because μ will be big). Only the lowest-order terms we can put in our Lagrangian will come with dimensionless couplings (or ones with units of mass to a positive power), so we only need bother with those at low energies. This feature of field theory allows for a dramatic simplification in considering all of the models we might want to examine.

As mentioned at the beginning of this section, general relativity itself is a classical field theory, in which the dynamical field is the metric tensor. It is nevertheless fair to think of GR as somehow different; for the most part other classical field theories rely on the existence of a pre-existing spacetime geometry, whereas in GR the geometry is determined by the equations of motion. (There are exceptions to this idea, called topological field theories, in which the metric makes no appearance.) Our task in the next few chapters is to explore the nature of curved geometries as characterized by the spacetime metric, before moving in Chapter 4 to putting these notions to work in constructing a theory of gravitation.

1.11 ■ EXERCISES

1. Consider an inertial frame S with coordinates $x^\mu = (t, x, y, z)$, and a frame S' with coordinates $x^{\mu'}$ related to S by a boost with velocity parameter v along the y-axis. Imagine we have a wall at rest in S', lying along the line $x' = -y'$. From the point of view of S, what is the relationship between the incident angle of a ball hitting the wall (traveling in the x-y plane) and the reflected angle? What about the velocity before and after?

2. Imagine that space (not spacetime) is actually a finite box, or in more sophisticated terms, a three-torus, of size L. By this we mean that there is a coordinate system $x^\mu = (t, x, y, z)$ such that every point with coordinates (t, x, y, z) is *identified* with every point with coordinates $(t, x + L, y, z)$, $(t, x, y + L, z)$, and $(t, x, y, z + L)$. Note that the time coordinate is the same. Now consider two observers; observer A is at rest in this coordinate system (constant spatial coordinates), while observer B moves in the x-direction with constant velocity v. A and B begin at the same event, and while A remains still, B moves once around the universe and comes back to intersect the worldline of A without ever having to accelerate (since the universe is periodic). What are the relative proper times experienced in this interval by A and B? Is this consistent with your understanding of Lorentz invariance?

3. Three events, A, B, C, are seen by observer $\mathcal{O}$ to occur in the order ABC. Another observer, $\bar{\mathcal{O}}$, sees the events to occur in the order CBA. Is it possible that a third observer sees the events in the order ACB? Support your conclusion by drawing a spacetime diagram.

4. Projection effects can trick you into thinking that an astrophysical object is moving "superluminally." Consider a quasar that ejects gas with speed v at an angle θ with respect to the line-of-sight of the observer. Projected onto the sky, the gas appears to travel perpendicular to the line of sight with angular speed v_{app}/D, where D is the distance to the quasar and v_{app} is the apparent speed. Derive an expression for v_{app} in terms of v and θ. Show that, for appropriate values of v and θ, v_{app} can be greater than 1.

5. Particle physicists are so used to setting $c = 1$ that they measure mass in units of energy. In particular, they tend to use electron volts ($1\ \text{eV} = 1.6 \times 10^{-12}\ \text{erg} = 1.8 \times 10^{-33}\ \text{g}$), or, more commonly, keV, MeV, and GeV (10^3 eV, 10^6 eV, and 10^9 eV, respectively). The muon has been measured to have a mass of 0.106 GeV and a rest frame lifetime of 2.19×10^{-6} seconds. Imagine that such a muon is moving in the circular storage ring of a particle accelerator, 1 kilometer in diameter, such that the muon's total energy is 1000 GeV. How long would it appear to live from the experimenter's point of view? How many radians would it travel around the ring?

6. In Euclidean three-space, let p be the point with coordinates $(x, y, z) = (1, 0, -1)$. Consider the following curves that pass through p:

$$x^i(\lambda) = (\lambda, (\lambda - 1)^2, -\lambda)$$
$$x^i(\mu) = (\cos\mu, \sin\mu, \mu - 1)$$
$$x^i(\sigma) = (\sigma^2, \sigma^3 + \sigma^2, \sigma).$$

   (a) Calculate the components of the tangent vectors to these curves at p in the coordinate basis $\{\partial_x, \partial_y, \partial_z\}$.

   (b) Let $f = x^2 + y^2 - yz$. Calculate $df/d\lambda$, $df/d\mu$ and $df/d\sigma$.

7. Imagine we have a tensor $X^{\mu\nu}$ and a vector V^μ, with components

$$X^{\mu\nu} = \begin{pmatrix} 2 & 0 & 1 & -1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{pmatrix}, \qquad V^\mu = (-1, 2, 0, -2).$$

Find the components of:

(a) $X^{\mu}{}_{\nu}$

(b) $X_{\mu}{}^{\nu}$

(c) $X^{(\mu\nu)}$

(d) $X_{[\mu\nu]}$

(e) $X^{\lambda}{}_{\lambda}$

(f) $V^{\mu}V_{\mu}$

(g) $V_{\mu}X^{\mu\nu}$

8. If $\partial_{\nu}T^{\mu\nu} = Q^{\mu}$, what physically does the spatial vector Q^{i} represent? Use the dust energy momentum tensor to make your case.

9. For a system of discrete point particles the energy-momentum tensor takes the form

$$T_{\mu\nu} = \sum_{a} \frac{p_{\mu}^{(a)} p_{\nu}^{(a)}}{p^{0(a)}} \delta^{(3)}(\mathbf{x} - \mathbf{x}^{(a)}), \tag{1.173}$$

where the index a labels the different particles. Show that, for a dense collection of particles with isotropically distributed velocities, we can smooth over the individual particle worldlines to obtain the perfect-fluid energy-momentum tensor (1.114).

10. Using the tensor transformation law applied to $F_{\mu\nu}$, show how the electric and magnetic field 3-vectors **E** and **B** transform under

(a) a rotation about the y-axis,

(b) a boost along the z-axis.

11. Verify that (1.98) is indeed equivalent to (1.97), and that they are both equivalent to the last two equations in (1.93).

12. Consider the two field theories we explicitly discussed, Maxwell's electromagnetism (let $J^{\mu} = 0$) and the scalar field theory defined by (1.148).

(a) Express the components of the energy-momentum tensors of each theory in three-vector notation, using the divergence, gradient, curl, electric, and magnetic fields, and an overdot to denote time derivatives.

(b) Using the equations of motion, verify (in any notation you like) that the energy-momentum tensors are conserved.

13. Consider adding to the Lagrangian for electromagnetism an additional term of the form $\mathcal{L}' = \tilde{\epsilon}_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}$.

(a) Express $\mathcal{L}'$ in terms of **E** and **B**.

(b) Show that including $\mathcal{L}'$ does not affect Maxwell's equations. Can you think of a deep reason for this?

CHAPTER

2 Manifolds

2.1 ■ GRAVITY AS GEOMETRY

Gravity is special. In the context of general relativity, we ascribe this specialness to the fact that the dynamical field giving rise to gravitation is the metric tensor describing the curvature of spacetime itself, rather than some additional field propagating through spacetime; this was Einstein's profound insight. The physical principle that led him to this idea was the *universality* of the gravitational interaction, as formalized by the **Principle of Equivalence**. Let's see how this physical principle leads us to the mathematical strategy of describing gravity as the geometry of a curved manifold.

The Principle of Equivalence comes in a variety of forms, the first of which is the **Weak Equivalence Principle**, or WEP. The WEP states that the inertial mass and gravitational mass of any object are equal. To see what this means, think about Newtonian mechanics. The Second Law relates the force exerted on an object to the acceleration it undergoes, setting them proportional to each other with the constant of proportionality being the inertial mass m_i:

$$\mathbf{F} = m_i \mathbf{a}. \tag{2.1}$$

The inertial mass clearly has a universal character, related to the resistance you feel when you try to push on the object; it takes the same value no matter what kind of force is being exerted. We also have Newton's law of gravitation, which can be thought of as stating that the gravitational force exerted on an object is proportional to the gradient of a scalar field Φ, known as the gravitational potential. The constant of proportionality in this case is called the gravitational mass m_g:

$$\mathbf{F}_g = -m_g \nabla \Phi. \tag{2.2}$$

On the face of it, m_g has a very different character than m_i; it is a quantity specific to the gravitational force. If you like, m_g/m_i can be thought of as the "gravitational charge" of the body. Nevertheless, Galileo long ago showed (apocryphally by dropping weights off of the Leaning Tower of Pisa, actually by rolling balls down inclined planes) that the response of matter to gravitation is universal—every object falls at the same rate in a gravitational field, independent of the composition of the object. In Newtonian mechanics this translates into the WEP, which is simply

$$m_i = m_g \tag{2.3}$$

for any object. An immediate consequence is that the behavior of freely-falling test particles is universal, independent of their mass (or any other qualities they may have); in fact, we have

$$\mathbf{a} = -\nabla\Phi. \tag{2.4}$$

Experimentally, the independence of the acceleration due to gravity on the composition of the falling object has been verified to extremely high precision by the Eötvös experiment and its modern successors.

This suggests an equivalent formulation of the WEP: there exists a preferred class of trajectories through spacetime, known as *inertial* (or "freely-falling") trajectories, on which unaccelerated particles travel—where unaccelerated means "subject only to gravity." Clearly this is not true for other forces, such as electromagnetism. In the presence of an electric field, particles with opposite charges will move on quite different trajectories. Every particle, on the other hand, has an identical gravitational charge.

The universality of gravitation, as implied by the WEP, can be stated in another, more popular, form. Imagine that we consider a physicist in a tightly sealed box, unable to observe the outside world, who is doing experiments involving the motion of test particles, for example to measure the local gravitational field. Of course she would obtain different answers if the box were sitting on the moon or on Jupiter than she would on Earth. But the answers would also be different if the box were accelerating at a constant rate; this would change the acceleration of the freely-falling particles with respect to the box. The WEP implies that there is no way to disentangle the effects of a gravitational field from those of being in a uniformly accelerating frame, simply by observing the behavior of freely-falling particles. This follows from the universality of gravitation; in electrodynamics, in contrast, it would be possible to distinguish between uniform acceleration and an electromagnetic field, by observing the behavior of particles with different charges. But with gravity it is impossible, since the "charge" is necessarily proportional to the (inertial) mass.

To be careful, we should limit our claims about the impossibility of distinguishing gravity from uniform acceleration by restricting our attention to "small enough regions of spacetime." If the sealed box were sufficiently big, the gravitational field would change from place to place in an observable way, while the effect of acceleration would always be in the same direction. In a rocket ship or elevator, the particles would always fall straight down. In a very big box in a gravitational field, however, the particles would move toward the center of the Earth, for example, which would be a different direction for widely separated experiments. The WEP can therefore be stated as follows: *The motion of freely-falling particles are the same in a gravitational field and a uniformly accelerated frame, in small enough regions of spacetime.* In larger regions of spacetime there will be inhomogeneities in the gravitational field, which will lead to tidal forces, which can be detected.

After the advent of special relativity, the concept of mass lost some of its uniqueness, as it became clear that mass was simply a manifestation of energy

and momentum (as we have seen in Chapter 1). It was therefore natural for Einstein to think about generalizing the WEP to something more inclusive. His idea was simply that there should be no way whatsoever for the physicist in the box to distinguish between uniform acceleration and an external gravitational field, no matter what experiments she did (not only by dropping test particles). This reasonable extrapolation became what is now known as the **Einstein Equivalence Principle**, or EEP: *In small enough regions of spacetime, the laws of physics reduce to those of special relativity; it is impossible to detect the existence of a gravitational field by means of local experiments.*

In fact, it is hard to imagine theories that respect the WEP but violate the EEP. Consider a hydrogen atom, a bound state of a proton and an electron. Its mass is actually less than the sum of the masses of the proton and electron considered individually, because there is a negative binding energy—you have to put energy into the atom to separate the proton and electron. According to the WEP, the gravitational mass of the hydrogen atom is therefore less than the sum of the masses of its constituents; the gravitational field couples to electromagnetism (which holds the atom together) in exactly the right way to make the gravitational mass come out right. This means that not only must gravity couple to rest mass universally, but also to all forms of energy and momentum—which is practically the claim of the EEP. It is possible to come up with counterexamples, however; for example, we could imagine a theory of gravity in which freely falling particles began to rotate as they moved through a gravitational field. Then they could fall along the same paths as they would in an accelerated frame (thereby satisfying the WEP), but you could nevertheless detect the existence of the gravitational field (in violation of the EEP). Such theories seem contrived, but there is no law of nature that forbids them.

Sometimes a distinction is drawn between "gravitational laws of physics" and "nongravitational laws of physics," and the EEP is defined to apply only to the latter. Then the Strong Equivalence Principle (SEP) is defined to include all of the laws of physics, gravitational and otherwise. A theory that violated the SEP but not the EEP would be one in which the *gravitational* binding energy did not contribute equally to the inertial and gravitational mass of a body; thus, for example, test particles with appreciable self-gravity (to the extent that such a concept makes sense) could fall along different trajectories than lighter particles.

It is the EEP that implies (or at least suggests) that we should attribute the action of gravity to the curvature of spacetime. Remember that in special relativity a prominent role is played by inertial frames—while it is not possible to single out some frame of reference as uniquely "at rest," it is possible to single out a family of frames that are "unaccelerated" (inertial). The acceleration of a charged particle in an electromagnetic field is therefore uniquely defined with respect to these frames. The EEP, on the other hand, implies that gravity is inescapable—there is no such thing as a "gravitationally neutral object" with respect to which we can measure the acceleration due to gravity. It follows that the acceleration due to gravity is not something that can be reliably defined, and therefore is of little use.

Instead, it makes more sense to *define* "unaccelerated" as "freely falling," and that is what we shall do. From here we are led to the idea that gravity is not a "force"—a force is something that leads to acceleration, and our definition of zero acceleration is "moving freely in the presence of whatever gravitational field happens to be around."

This seemingly innocuous step has profound implications for the nature of spacetime. In SR, we have a procedure for starting at some point and constructing an inertial frame that stretches throughout spacetime, by joining together rigid rods and attaching clocks to them. But, again due to inhomogeneities in the gravitational field, this is no longer possible. If we start in some freely-falling state and build a large structure out of rigid rods, at some distance away freely-falling objects will look like they are accelerating with respect to this reference frame, as shown in Figure 2.1. The solution is to retain the notion of inertial frames, but to discard the hope that they can be uniquely extended throughout space and time. Instead we can define **locally inertial frames**, those that follow the motion of individual freely falling particles in small enough regions of spacetime. (Every time we say "small enough regions," purists should imagine a limiting procedure in which we take the appropriate spacetime volume to zero.) This is the best we can do, but it forces us to give up a good deal. For example, we can no longer speak with confidence about the relative velocity of far-away objects, since the inertial reference frames appropriate to those objects are completely different from those appropriate to us.

Our job as physicists is to construct mathematical models of the world, and then test the predictions of such models against observations and experiments. Following the implications of the universality of gravitation has led us to give up on the idea of expressing gravity as a force propagating through spacetime,

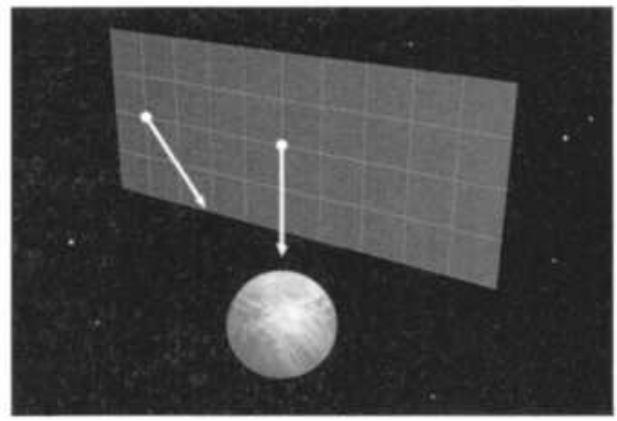


FIGURE 2.1 Failure of global frames. Since every particle feels the influence of gravity, we define "unaccelerating" as "freely falling." As a consequence, it becomes impossible to define globally inertial coordinate systems by the procedure outlined in Chapter 1, since particles initially at rest will begin to move with respect to such a frame.

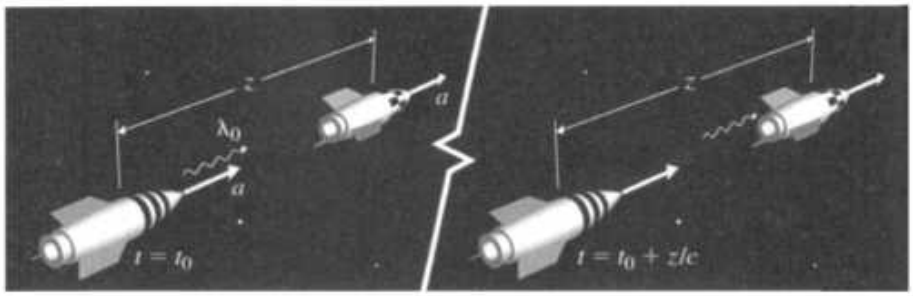


FIGURE 2.2 The Doppler shift as measured by two rockets separated by a distance z, each feeling an acceleration a.

and indeed to give up on the idea of global reference frames stretching throughout spacetime. We therefore need to invoke a mathematical framework in which physical theories can be consistent with these conclusions. The solution will be to imagine that spacetime has a curved geometry, and that gravitation is a manifestation of this curvature. The appropriate mathematical structure used to describe curvature is that of a *differentiable manifold*: essentially, a kind of set that looks locally like flat space, but might have a very different global geometry. (Remember that the EEP can be stated as "the laws of physics reduce to those of special relativity in small regions of spacetime," which matches well with the mathematical notion of a set that locally resembles flat space.)

We cannot prove that gravity should be thought of as the curvature of spacetime; instead we can propose the idea, derive its consequences, and see if the result is a reasonable fit to our experience of the world. Let's set about doing just that.

FIGURE 2.3 Gravitational redshift on the surface of the Earth, as measured by observers at different elevations.

Consider one of the celebrated predictions of the EEP, the gravitational redshift. Imagine two rockets, a distance z apart, each moving with some constant acceleration a in a region far away from any gravitational fields, as shown in Figure 2.2. At time t_0 the trailing rocket emits a photon of wavelength λ_0. The rockets remain a constant distance apart, so the photon reaches the leading rocket after a time $\Delta t = z/c$ in our background reference frame. (We assume $\Delta v/c$ is small, so we only work to first order.) In this time the rockets will have picked up an additional velocity $\Delta v = a\Delta t = az/c$. Therefore, the photon reaching the leading rocket will be redshifted by the conventional Doppler effect, by an amount

$$\frac{\Delta\lambda}{\lambda_0} = \frac{\Delta v}{c} = \frac{az}{c^2}. \tag{2.5}$$

According to the EEP, the same thing should happen in a uniform gravitational field. So we imagine a tower of height z sitting on the surface of a planet, with a_g the strength of the gravitational field (what Newton would have called the "acceleration due to gravity"), as portrayed in Figure 2.3. We imagine that observers in the rocket at the top of the tower are able to detect photons emitted from the ground, but are otherwise unable to look outside and see that they are sitting on a

tower. In other words, they have no way of distinguishing this situation from that of the accelerating rockets. Therefore, the EEP allows us to conclude immediately that a photon emitted from the ground with wavelength λ_0 will be redshifted by an amount

$$\frac{\Delta\lambda}{\lambda_0} = \frac{a_g z}{c^2}. \tag{2.6}$$

This is the famous gravitational redshift. Notice that it is a direct consequence of the EEP; the details of general relativity were not required.

The formula for the redshift is more often stated in terms of the Newtonian potential Φ, where $\mathbf{a}_g = \nabla\Phi$. (The sign is changed with respect to the usual convention, since we are thinking of $\mathbf{a}_g$ as the acceleration of the reference frame, not of a particle with respect to this reference frame.) A nonconstant gradient of Φ is like a time-varying acceleration, and the equivalent net velocity is given by integrating over the time between emission and absorption of the photon. We then have

$$\begin{aligned}\frac{\Delta\lambda}{\lambda_0} &= \frac{1}{c}\int \nabla\Phi\, dt \\ &= \frac{1}{c^2}\int \partial_z\Phi\, dz \\ &= \Delta\Phi,\end{aligned} \tag{2.7}$$

where $\Delta\Phi$ is the total change in the gravitational potential, and we have once again set $c = 1$. This simple formula for the gravitational redshift continues to be true in more general circumstances. Of course, by using the Newtonian potential at all, we are restricting our domain of validity to weak gravitational fields.

From the EEP we have argued in favor of a gravitational redshift; we may now use this phenomenon to provide further support for the idea that we should think of spacetime as curved. Consider the same experimental setup that we had before, now portrayed on the spacetime diagram in Figure 2.4. A physicist on the ground emits a beam of light with wavelength λ_0 from a height z_0, which travels to the top of the tower at height z_1. The time between when the beginning of any single wavelength of the light is emitted and the end of that same wavelength is emitted is $\Delta t_0 = \lambda_0/c$, and the same time interval for the absorption is $\Delta t_1 = \lambda_1/c$, where time is measured by clocks located at the respective elevations. Since we imagine that the gravitational field is static, the paths through spacetime followed by the leading and trailing edge of the single wave must be precisely congruent. (They are represented by generic curved paths, since we do not pretend that we know just what the paths will be.) Simple geometry seems to imply that the times Δt_0 and Δt_1 must be the same. But of course they are not; the gravitational redshift implies that the elevated experimenters observe fewer wavelengths per second, so that $\Delta t_1 > \Delta t_0$. We can interpret this roughly as "the clock on the tower appears to run more quickly." What went wrong? Simple geometry—the spacetime through which the photons traveled was curved.

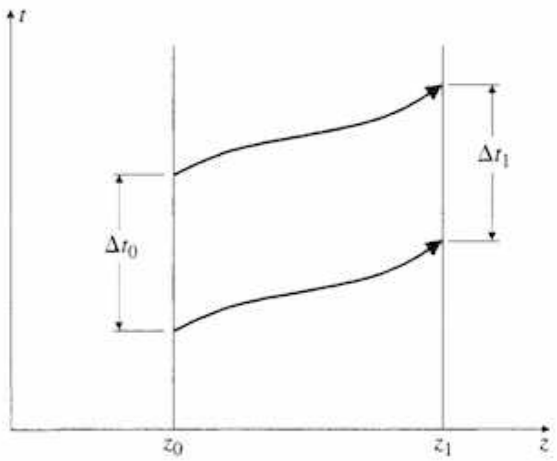


FIGURE 2.4 Spacetime diagram of the gravitational-redshift experiment portrayed in Figure 2.3. Spacetime paths beginning at different moments are congruent, but the time intervals as measured on the ground and on the tower are different, signaling a breakdown of Euclidean geometry.

We therefore would like to describe spacetime as a kind of mathematical structure that looks locally like Minkowski space, but may possess nontrivial curvature over extended regions. The kind of object that encompasses this notion is that of a manifold. In this chapter we will confine ourselves to understanding the concept of manifolds and the structures we may define on them, leaving the precise characterization of curvature for the next chapter.

2.2 ■ WHAT IS A MANIFOLD?

Manifolds (or differentiable manifolds) are one of the most fundamental concepts in mathematics and physics. We are all used to the properties of n-dimensional Euclidean space, $\mathbf{R}^n$, the set of n-tuples $(x^1, \ldots, x^n)$, often equipped with a flat positive-definite metric with components δ_{ij}. Mathematicians have worked for many years to develop the theory of analysis in $\mathbf{R}^n$—differentiation, integration, properties of functions, and so on. But clearly there are other spaces (spheres, for example) which we intuitively think of as "curved" or perhaps topologically complicated, on which we would like to perform analogous operations.

To address this problem we invent the notion of a manifold, which corresponds to a space that may be curved and have a complicated topology, but in local regions looks just like $\mathbf{R}^n$. Here by "looks like" we do not mean that the metric is the same, but only that more primitive notions like functions and coordinates work in a similar way. The entire manifold is constructed by smoothly sewing together these local regions. A crucial point is that the dimensionality n of the Euclidean spaces being used must be the same in every patch of the manifold; we then say

that the manifold is of dimension n. With this approach we can analyze functions on such a space by converting them (locally) to functions in a Euclidean space. Examples of manifolds include:

- $\mathbf{R}^n$ itself, including the line ($\mathbf{R}$), the plane ($\mathbf{R}^2$), and so on. This should be obvious, since $\mathbf{R}^n$ looks like $\mathbf{R}^n$ not only locally but globally.
- The n-sphere, S^n. This can be defined as the locus of all points some fixed distance from the origin in $\mathbf{R}^{n+1}$. The circle is of course S^1, and the two-sphere S^2 is one of the most useful examples of a manifold. (The zero-sphere S^0, if you think about it, consists of two points. We say that S^0 is a disconnected zero-dimensional manifold.) It's worth emphasizing that the definition of S^n in terms of an embedding in $\mathbf{R}^{n+1}$ is simply a convenient shortcut; all of the manifolds we will discuss may be defined in their own right, without recourse to higher-dimensional flat spaces.
- The n-torus T^n results from taking an n-dimensional cube and identifying opposite sides. The two-torus T^2 is a square with opposite sides identified, as shown in Figure 2.5. The surface of a doughnut is a familiar example.
- A Riemann surface of genus g is essentially a two-torus with g holes instead of just one, as shown in Figure 2.6. S^2 may be thought of as a Riemann surface of genus zero. In technical terms (not really relevant to our present dis-

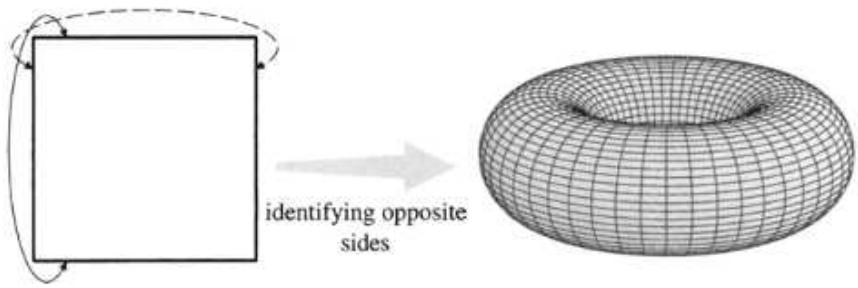


FIGURE 2.5 The torus, T^2, constructed by identifying opposite sides of a square.

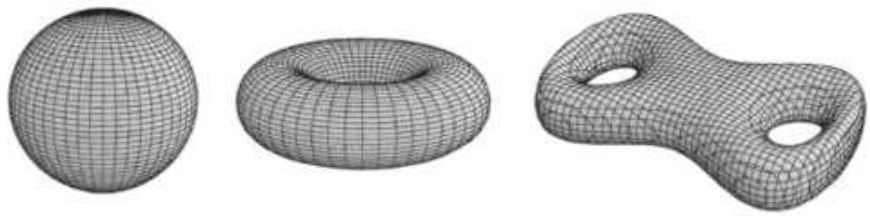


FIGURE 2.6 Riemann surfaces of different genera (plural of "genus").

cussion), every "compact orientable boundaryless" two-dimensional manifold is a Riemann surface of some genus.

- More abstractly, a set of continuous transformations such as rotations in $\mathbf{R}^n$ forms a manifold. Lie groups are manifolds that also have a group structure. So for example SO(2), the set of rotations in two dimensions, is the same manifold as S^1 (although in general group manifolds will be more complicated than spheres).

- The direct product of two manifolds is a manifold. That is, given manifolds M and M' of dimension n and n', we can construct a manifold $M \times M'$, of dimension $n + n'$, consisting of ordered pairs (p, p') with $p \in M$ and $p' \in M'$.

With all of these examples, the notion of a manifold may seem vacuous: what *isn't* a manifold? Plenty of things are not manifolds, because somewhere they do not look locally like $\mathbf{R}^n$. Examples include a one-dimensional line running into a two-dimensional plane, and two cones stuck together at their vertices, as portrayed in Figure 2.7. More subtle examples are shown in Figure 2.8. Consider for example a single (two-dimensional) cone. There is clearly a sense in which the cone looks locally like $\mathbf{R}^2$; at the same time, there is just as clearly something singular about the vertex of the cone. This is where the word "differentiable" in "differentiable manifold" begins to play a role; as we will see when we develop the formal definition, the cone is perfectly smooth as a manifold, even though the curvature is not smooth at its vertex. (Other types of singularities are more severe, and will prevent us from thinking of certain spaces as manifolds, smooth or otherwise.) Another example is a line segment (with endpoints included). This

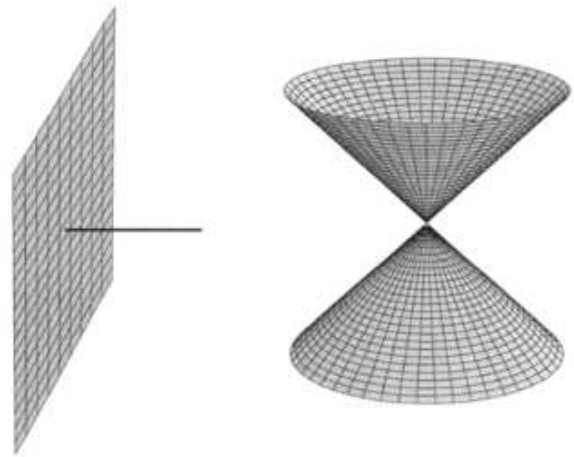


FIGURE 2.7 Examples of spaces that are not manifolds: a line ending on a plane, and two cones intersecting at their vertices. In each case there is a point that does not look locally like a Euclidean space of fixed dimension.

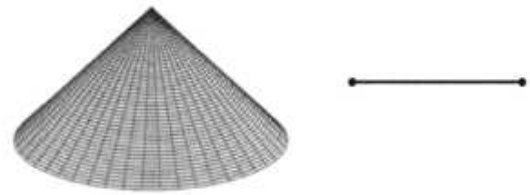


FIGURE 2.8 Subtle examples. The single cone is a smooth manifold, even though the curvature is singular at its vertex. A line segment is not a manifold, but may be described by the more general notion of "manifold with boundary."

certainly will not fit under the definition of manifolds we will develop, due to the endpoints. Nevertheless, we can extend the definition to include "manifolds with boundary," of which the line segment is a paradigmatic example. A brief discussion of manifolds with boundary is in Appendix D.

These subtle cases should convince you of the need for a rigorous definition, which we now begin to construct; our discussion follows that of Wald (1984). The informal idea of a manifold is that of a space consisting of patches that look locally like $\mathbf{R}^n$, and are smoothly sewn together. We therefore need to formalize the notions of "looking locally like $\mathbf{R}^n$" and "smoothly sewn together." We require a number of preliminary definitions, most of which are fairly clear, but it's nice to be complete. The most elementary notion is that of a **map** between two sets. (We assume you know what a set is, or think you do; we won't need to be too precise.) Given two sets M and N, a map $\phi : M \to N$ is a relationship that assigns, to each element of M, exactly one element of N. A map is therefore just a simple generalization of a function. Given two maps $\phi : A \to B$ and $\psi : B \to C$, we define the **composition** $\psi \circ \phi : A \to C$ by the operation $(\psi \circ \phi)(a) = \psi(\phi(a))$, as in Figure 2.9. So $a \in A$, $\phi(a) \in B$, and thus $(\psi \circ \phi)(a) \in C$. The order in which the maps are written makes sense, since the one on the right acts first.

A map ϕ is called **one-to-one** (or injective) if each element of N has at most one element of M mapped into it, and **onto** (or surjective) if each element of N has at least one element of M mapped into it. (If you think about it, better names for "one-to-one" would be "one-from-one" or for that matter "two-to-two.") Consider functions $\phi : \mathbf{R} \to \mathbf{R}$. Then $\phi(x) = e^x$ is one-to-one, but not onto; $\phi(x) = x^3 - x$ is onto, but not one-to-one; $\phi(x) = x^3$ is both; and $\phi(x) = x^2$ is neither, as in Figure 2.10.

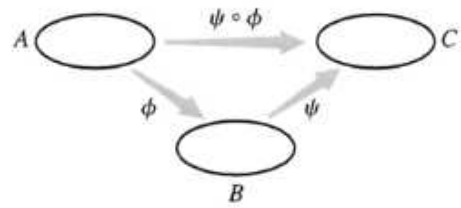
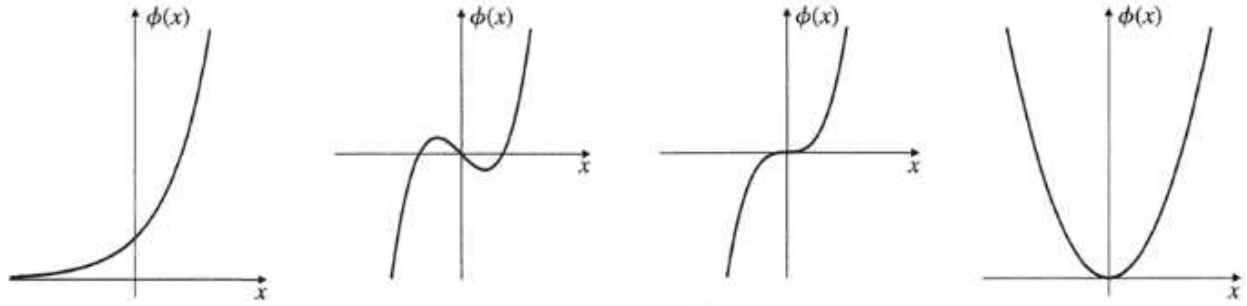


FIGURE 2.9 The map $\psi \circ \phi : A \to C$ is formed by composing $\phi : A \to B$ and $\psi : B \to C$.



one-to-one, not onto | onto, not one-to-one | both | neither

FIGURE 2.10 Types of maps.

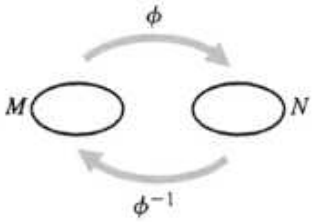


FIGURE 2.11 A map and its inverse.

The set M is known as the **domain** of the map ϕ, and the set of points in N that M gets mapped into is called the **image** of ϕ. For any subset $U \subset N$, the set of elements of M that get mapped to U is called the **preimage** of U under ϕ, or $\phi^{-1}(U)$. A map that is both one-to-one and onto is known as **invertible** (or bijective). In this case we can define the **inverse map** $\phi^{-1} : N \to M$ by $(\phi^{-1} \circ \phi)(a) = a$, as in Figure 2.11. Note that the same symbol ϕ^{-1} is used for both the preimage and the inverse map, even though the former is always defined and the latter is only defined in some special cases.

The notion of **continuity** of a map is actually a very subtle one, the precise formulation of which we won't need. Instead we will assume you understand the concepts of continuity and differentiability as applied to ordinary functions, maps $\phi : \mathbf{R} \to \mathbf{R}$. It will then be useful to extend these notions to maps between more general Euclidean spaces, $\phi : \mathbf{R}^m \to \mathbf{R}^n$. A map from $\mathbf{R}^m$ to $\mathbf{R}^n$ takes an m-tuple $(x^1, x^2, \ldots, x^m)$ to an n-tuple $(y^1, y^2, \ldots, y^n)$, and can therefore be thought of as a collection of n functions ϕ^i of m variables:

$$\begin{aligned} y^1 &= \phi^1(x^1, x^2, \ldots, x^m) \\ y^2 &= \phi^2(x^1, x^2, \ldots, x^m) \\ &\vdots \\ y^n &= \phi^n(x^1, x^2, \ldots, x^m). \end{aligned} \tag{2.8}$$

We will refer to any one of these functions as C^p if its pth derivative exists and is continuous, and refer to the entire map $\phi : \mathbf{R}^m \to \mathbf{R}^n$ as C^p if each of its component functions are at least C^p. Thus a C^0 map is continuous but not necessarily differentiable, while a C^∞ map is continuous and can be differentiated as many times as you like. Consider for example the function of one variable $\phi(x) = |x^3|$. This function is infinitely differentiable everywhere except at $x = 0$, where it is differentiable twice but not three times; we therefore say that it is C^2. C^∞ maps are sometimes called **smooth**.

We will call two sets M and N **diffeomorphic** if there exists a C^∞ map $\phi : M \to N$ with a C^∞ inverse $\phi^{-1} : N \to M$; the map ϕ is then called a diffeomorphism. This is the best notion we have that two spaces are "the same" as manifolds. For example, when we said that SO(2) was the same manifold as S^1, we meant they were diffeomorphic. See Appendix B for more discussion.

These basic definitions may have been familiar to you, even if only vaguely remembered. We will now put them to use in the rigorous definition of a manifold. Unfortunately, a somewhat baroque procedure is required to formalize this relatively intuitive notion. We will first have to define the notion of an open set, on which we can put coordinate systems, and then sew the open sets together in an appropriate way.

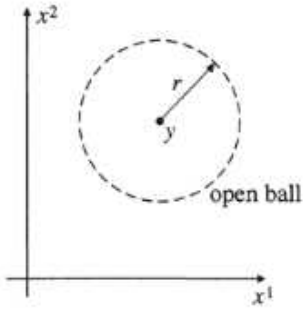


FIGURE 2.12 An open ball defined in $\mathbf{R}^n$.

We start with the notion of an **open ball**, which is the set of all points x in $\mathbf{R}^n$ such that $|x - y| < r$ for some fixed $y \in \mathbf{R}^n$ and $r \in \mathbf{R}$, where $|x - y| = [\sum_i (x^i - y^i)^2]^{1/2}$. Note that this is a strict inequality—the open ball is the interior of an n-sphere of radius r centered at y, as shown in Figure 2.12. An **open set** in $\mathbf{R}^n$ is a set constructed from an arbitrary (maybe infinite) union of open balls. In other words, $V \subset \mathbf{R}^n$ is open if, for any $y \in V$, there is an open ball centered at y that is completely inside V. Roughly speaking, an open set is the interior of some $(n-1)$-dimensional closed surface (or the union of several such interiors).

A **chart** or **coordinate system** consists of a subset U of a set M, along with a one-to-one map $\phi : U \to \mathbf{R}^n$, such that the image $\phi(U)$ is open in $\mathbf{R}^n$, as in Figure 2.13. (Any map is onto its image, so the map $\phi : U \to \phi(U)$ is invertible if it is one-to-one.) We then can say that U is an open set in M. A C^∞ **atlas** is an indexed collection of charts $\{(U_\alpha, \phi_\alpha)\}$ that satisfies two conditions:

1. The union of the U_α is equal to M; that is, the U_α cover M.
2. The charts are smoothly sewn together. More precisely, if two charts overlap, $U_\alpha \cap U_\beta \neq \emptyset$, then the map $(\phi_\alpha \circ \phi_\beta^{-1})$ takes points in $\phi_\beta(U_\alpha \cap U_\beta) \subset \mathbf{R}^n$ *onto* an open set $\phi_\alpha(U_\alpha \cap U_\beta) \subset \mathbf{R}^n$, and all of these maps must be C^∞ where they are defined. This should be clearer from Figure 2.14, adapted from Wald (1984).

So a chart is what we normally think of as a coordinate system on some open set, and an atlas is a system of charts that are smoothly related on their overlaps.

At long last, then: a C^∞ n-dimensional **manifold** (or n-manifold for short) is simply a set M along with a maximal atlas, one that contains every possible

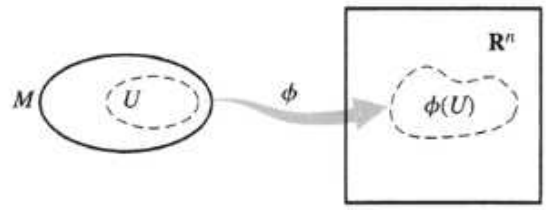


FIGURE 2.13 A coordinate chart covering an open subset U of M.

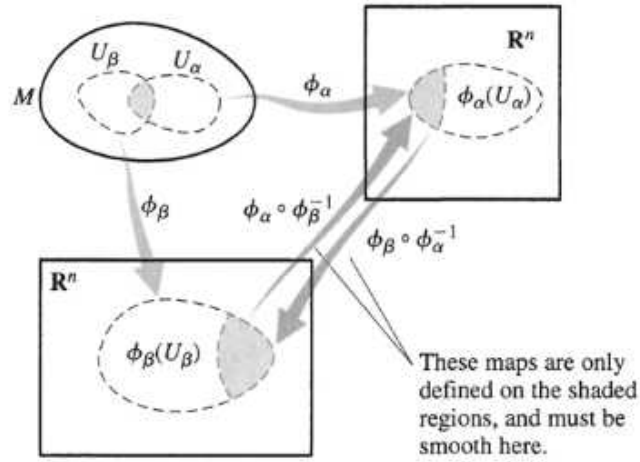


FIGURE 2.14 Overlapping coordinate charts.

compatible chart. We can also replace C^∞ by C^p in all the above definitions. For our purposes the degree of differentiability of a manifold is not crucial; we will always assume that any manifold is as differentiable as necessary for the application under consideration. The requirement that the atlas be maximal is so that two equivalent spaces equipped with different atlases don't count as different manifolds. This definition captures in formal terms our notion of a set that looks locally like $\mathbf{R}^n$. Of course we will rarely have to make use of the full power of the definition, but precision is its own reward.

One nice thing about our definition is that it does not rely on an embedding of the manifold in some higher-dimensional Euclidean space. In fact, any n-dimensional manifold can be embedded in $\mathbf{R}^{2n}$ (Whitney's embedding theorem), and sometimes we will make use of this fact, such as in our definition of the sphere above. (A Klein bottle is an example of a 2-manifold that cannot be embedded in $\mathbf{R}^3$, although it can be embedded in $\mathbf{R}^4$.) But it is important to recognize that the manifold has an individual existence independent of any embedding. It is not necessary to believe, for example, that four-dimensional spacetime is stuck in some larger space. On the other hand, it might be; we really don't know. Recent advances in string theory have led to the suggestion that our visible universe is actually a "brane" (generalization of "membrane") inside a higher-dimensional space. But as far as classical GR is concerned, the four-dimensional view is perfectly adequate.

Why was it necessary to be so finicky about charts and their overlaps, rather than just covering every manifold with a single chart? Because most manifolds cannot be covered with just one chart. Consider the simplest example, S^1. There is a conventional coordinate system, $\theta : S^1 \to \mathbf{R}$, where $\theta = 0$ at the top of the circle and wraps around to 2π. However, in the definition of a chart we have required that the image $\theta(S^1)$ be open in $\mathbf{R}$. If we include either $\theta = 0$ or $\theta = 2\pi$, we have a closed interval rather than an open one; if we exclude both points, we

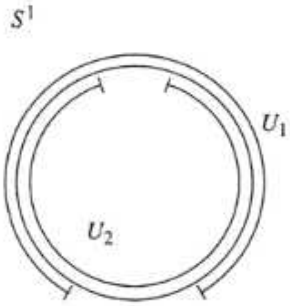


FIGURE 2.15 Two coordinate charts, which together cover S^1.

haven't covered the whole circle. So we need at least two charts, as shown in Figure 2.15.

A somewhat more complicated example is provided by S^2, where once again no single chart will cover the manifold. A Mercator projection, traditionally used for world maps, misses both the North and South poles (as well as the International Date Line, which involves the same problem with θ that we found for S^1.) Let's take S^2 to be the set of points in $\mathbf{R}^3$ defined by $(x^1)^2 + (x^2)^2 + (x^3)^2 = 1$. We can construct a chart from an open set U_1, defined to be the sphere minus the north pole, via stereographic projection, illustrated in Figure 2.16. Thus, we draw a straight line from the north pole to the plane defined by $x^3 = -1$, and assign to the point on S^2 intercepted by the line the Cartesian coordinates (y^1, y^2) of the appropriate point on the plane. Explicitly, the map is given by

$$\phi_1(x^1, x^2, x^3) \equiv (y^1, y^2) = \left(\frac{2x^1}{1 - x^3}, \frac{2x^2}{1 - x^3} \right). \tag{2.9}$$

Check this for yourself. Another chart (U_2, ϕ_2) is obtained by projecting from the south pole to the plane defined by $x^3 = +1$. The resulting coordinates cover the sphere minus the south pole, and are given by

$$\phi_2(x^1, x^2, x^3) \equiv (z^1, z^2) = \left(\frac{2x^1}{1 + x^3}, \frac{2x^2}{1 + x^3} \right). \tag{2.10}$$

Together, these two charts cover the entire manifold, and they overlap in the region $-1 < x^3 < +1$. Another thing you can check is that the composition $\phi_2 \circ \phi_1^{-1}$ is given by

$$z^i = \frac{4y^i}{[(y^1)^2 + (y^2)^2]}, \tag{2.11}$$

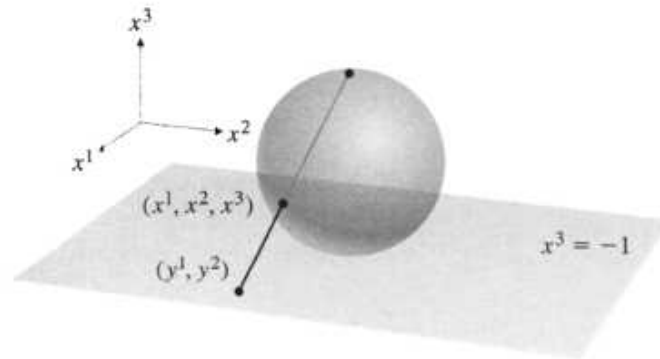


FIGURE 2.16 Defining a stereographic coordinate chart on S^2 by projecting from the north pole down to a plane tangent to the south pole. Such a chart covers all of the sphere except for the north pole itself.

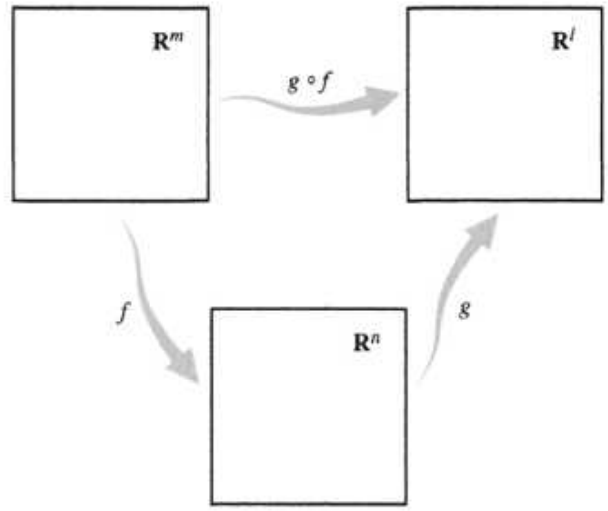


FIGURE 2.17 The chain rule relates the partial derivatives of $g \circ f$ to those of g and f.

and is C^∞ in the region of overlap. As long as we restrict our attention to this region, (2.11) is just what we normally think of as a change of coordinates.

We therefore see the necessity of charts and atlases: Many manifolds cannot be covered with a single coordinate system. Nevertheless, it is often convenient to work with a single chart, and just keep track of the set of points that aren't included.

One piece of conventional calculus that we will need later is the **chain rule**. Let us imagine that we have maps $f : \mathbf{R}^m \to \mathbf{R}^n$ and $g : \mathbf{R}^n \to \mathbf{R}^l$, and therefore the composition $(g \circ f) : \mathbf{R}^m \to \mathbf{R}^l$, as shown in Figure 2.17. We can label points in each space in terms of components: x^a on $\mathbf{R}^m$, y^b on $\mathbf{R}^n$, and z^c on $\mathbf{R}^l$, where the indices range over the appropriate values. The chain rule relates the partial derivatives of the composition to the partial derivatives of the individual maps:

$$\frac{\partial}{\partial x^a}(g \circ f)^c = \sum_b \frac{\partial f^b}{\partial x^a}\frac{\partial g^c}{\partial y^b}. \tag{2.12}$$

This is usually abbreviated to

$$\frac{\partial}{\partial x^a} = \sum_b \frac{\partial y^b}{\partial x^a}\frac{\partial}{\partial y^b}. \tag{2.13}$$

There is nothing illegal or immoral about using this shorthand form of the chain rule, but you should be able to visualize the maps that underlie the construction. Recall that when $m = n$, the determinant of the matrix $\partial y^b/\partial x^a$ is called the **Jacobian** of the map, and the map is invertible whenever the Jacobian is nonzero.

2.3 ■ VECTORS AGAIN

Having constructed this groundwork, we can now proceed to introduce various kinds of structure on manifolds. We begin with vectors and tangent spaces. In our discussion of special relativity we were intentionally vague about the definition of vectors and their relationship to the spacetime. One point we stressed was the notion of a tangent space—the set of all vectors at a single point in spacetime. The reason for this emphasis was to remove from your minds the idea that a vector stretches from one point on the manifold to another, but instead is just an object associated with a single point. What is temporarily lost by adopting this view is a way to make sense of statements like "the vector points in the x direction"—if the tangent space is merely an abstract vector space associated with each point, it's hard to know what this should mean. Now it's time to fix the problem.

Let's imagine that we wanted to construct the tangent space at a point p in a manifold M, using only things that are intrinsic to M (no embeddings in higher-dimensional spaces). A first guess might be to use our intuitive knowledge that there are objects called "tangent vectors to curves," which belong in the tangent space. We might therefore consider the set of all parameterized curves through p—that is, the space of all (nondegenerate) maps $\gamma : \mathbf{R} \to M$, such that p is in the image of γ. The temptation is to define the tangent space as simply the space of all tangent vectors to these curves at the point p. But this is obviously cheating; the tangent space T_p is supposed to be the space of vectors at p, and before we have defined this we don't have an independent notion of what "the tangent vector to a curve" is supposed to mean. In some coordinate system x^μ any curve through p defines an element of $\mathbf{R}^n$ specified by the n real numbers $dx^\mu/d\lambda$ (where λ is the parameter along the curve), but this map is clearly coordinate-dependent, which is not what we want.

Nevertheless we are on the right track, we just have to make things independent of coordinates. To this end we define $\mathcal{F}$ to be the space of all smooth functions on M (that is, C^∞ maps $f : M \to \mathbf{R}$). Then we notice that each curve through p defines an operator on this space, the directional derivative, which maps $f \to df/d\lambda$ (at p). We will make the following claim: *the tangent space T_p can be identified with the space of directional derivative operators along curves through p.* To establish this idea we must demonstrate two things: first, that the space of directional derivatives is a vector space, and second that it is the vector space we want (it has the same dimensionality as M, yields a natural idea of a vector pointing along a certain direction, and so on).

The first claim, that directional derivatives form a vector space, seems straightforward enough. Imagine two operators $d/d\lambda$ and $d/d\eta$ representing derivatives along two curves $x^\mu(\lambda)$ and $x^\mu(\eta)$ through p. There is no problem adding these and scaling by real numbers, to obtain a new operator $a(d/d\lambda) + b(d/d\eta)$. It is not immediately obvious, however, that the space closes; in other words, that the resulting operator is itself a derivative operator. A good derivative operator is one that acts linearly on functions, and obeys the conventional Leibniz (product) rule on products of functions. Our new operator is manifestly linear, so we need to

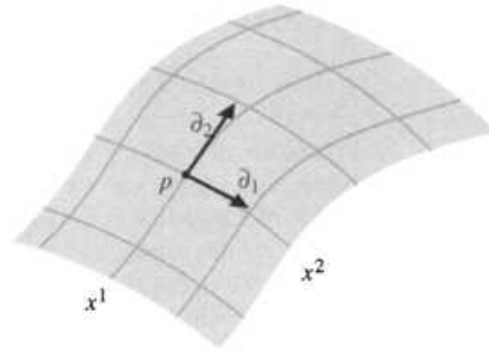


FIGURE 2.18 Partial derivatives define directional derivatives along curves that keep all of the other coordinates constant.

verify that it obeys the Leibniz rule. We have

$$\left(a\frac{d}{d\lambda}+b\frac{d}{d\eta}\right)(fg)=af\frac{dg}{d\lambda}+ag\frac{df}{d\lambda}+bf\frac{dg}{d\eta}+bg\frac{df}{d\eta}$$
$$=\left(a\frac{df}{d\lambda}+b\frac{df}{d\eta}\right)g+\left(a\frac{dg}{d\lambda}+b\frac{dg}{d\eta}\right)f. \quad (2.14)$$

As we had hoped, the product rule is satisfied, and the set of directional derivatives is therefore a vector space.

Is it the vector space that we would like to identify with the tangent space? The easiest way to become convinced is to find a basis for the space. Consider again a coordinate chart with coordinates x^μ. Then there is an obvious set of n directional derivatives at p, namely the partial derivatives ∂_μ at p, as shown in Figure 2.18. Note that this is really the *definition* of the partial derivative with respect to x^μ: the directional derivative along a curve defined by x^ν = constant for all $\nu \neq \mu$, parameterized by x^μ itself. We are now going to claim that the partial derivative operators $\{\partial_\mu\}$ at p form a basis for the tangent space T_p. (It follows immediately that T_p is n-dimensional, since that is the number of basis vectors.) To see this we will show that any directional derivative can be decomposed into a sum of real numbers times partial derivatives. This will just be the familiar expression for the components of a tangent vector, but it's nice to see it from the big-machinery approach. Consider an n-manifold M, a coordinate chart $\phi : M \to \mathbf{R}^n$, a curve $\gamma : \mathbf{R} \to M$, and a function $f : M \to \mathbf{R}$. This leads to the tangle of maps shown in Figure 2.19. If λ is the parameter along γ, we want to express the vector/operator $d/d\lambda$ in terms of the partials ∂_μ. Using the chain rule (2.12), we have

$$\frac{d}{d\lambda}f=\frac{d}{d\lambda}(f\circ\gamma)$$
$$=\frac{d}{d\lambda}[(f\circ\phi^{-1})\circ(\phi\circ\gamma)]$$

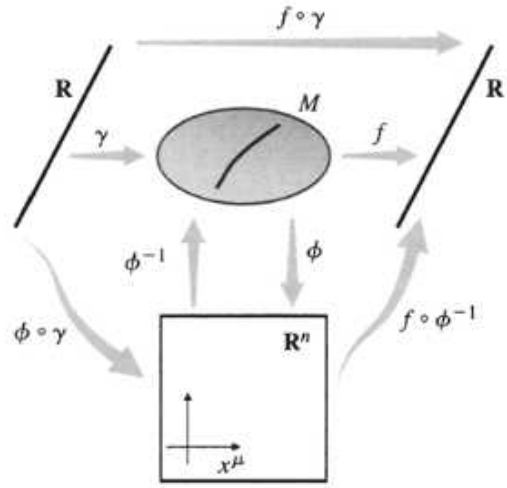


FIGURE 2.19 Decomposing the tangent vector to a curve $\gamma : \mathbf{R} \to M$ in terms of partial derivatives with respect to coordinates on M.

$$= \frac{d(\phi \circ \gamma)^\mu}{d\lambda} \frac{\partial (f \circ \phi^{-1})}{\partial x^\mu}$$

$$= \frac{dx^\mu}{d\lambda} \partial_\mu f. \tag{2.15}$$

The first line simply takes the informal expression on the left-hand side and rewrites it as an honest derivative of the function $(f \circ \gamma) : \mathbf{R} \to \mathbf{R}$. The second line just comes from the definition of the inverse map ϕ^{-1} (and associativity of the operation of composition). The third line is the formal chain rule (2.12), and the last line is a return to the informal notation of the start. Since the function f was arbitrary, we have

$$\frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda} \partial_\mu. \tag{2.16}$$

Thus, the partials $\{\partial_\mu\}$ do indeed represent a good basis for the vector space of directional derivatives, which we can therefore safely identify with the tangent space.

Of course, the vector represented by $d/d\lambda$ is one we already know; it's the tangent vector to the curve with parameter λ. Thus (2.16) can be thought of as a restatement of equation (1.38), where we claimed that the components of the tangent vector were simply $dx^\mu/d\lambda$. The only difference is that we are working on an arbitrary manifold, and we have specified our basis vectors to be $\hat{e}_{(\mu)} = \partial_\mu$.

This particular basis ($\hat{e}_{(\mu)} = \partial_\mu$) is known as a **coordinate basis** for T_p; it is the formalization of the notion of setting up the basis vectors to point along the coordinate axes. There is no reason why we are limited to coordinate bases when we consider tangent vectors. For example, the coordinate basis vectors are typically not normalized to unity, nor orthogonal to each other, as we shall see

shortly. This is not a situation we can define away; on a curved manifold, a coordinate basis will *never* be orthonormal throughout a neighborhood of any point where the curvature does not vanish. Of course we can define noncoordinate orthonormal bases, for example by giving their components in a coordinate basis, and sometimes this technique is useful. However, coordinate bases are very simple and natural, and we will use them almost exclusively throughout the book; for a look at orthonormal bases, see Appendix J. (It is standard in the study of vector analysis in three-dimensional Euclidean space to choose orthonormal bases rather than coordinate bases; you should therefore be careful when applying formulae from GR texts to the study of non-Cartesian coordinates in flat space.)

One of the advantages of the rather abstract point of view we have taken toward vectors is that the transformation law under changes of coordinates is immediate. Since the basis vectors are $\hat{e}_{(\mu)} = \partial_\mu$, the basis vectors in some new coordinate system $x^{\mu'}$ are given by the chain rule (2.13) as

$$\partial_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu . \tag{2.17}$$

We can get the transformation law for vector components by the same technique used in flat space, demanding that the vector $V = V^\mu \partial_\mu$ be unchanged by a change of basis. We have

$$\begin{aligned} V^\mu \partial_\mu &= V^{\mu'} \partial_{\mu'} \\ &= V^{\mu'} \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu , \end{aligned} \tag{2.18}$$

and hence, since the matrix $\partial x^{\mu'}/\partial x^\mu$ is the inverse of the matrix $\partial x^\mu / \partial x^{\mu'}$,

$$V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} V^\mu . \tag{2.19}$$

Since the basis vectors are usually not written explicitly, the rule (2.19) for transforming components is what we call the "vector transformation law." We notice that it is compatible with the transformation of vector components in special relativity under Lorentz transformations, $V^{\mu'} = \Lambda^{\mu'}{}_\mu V^\mu$, since a Lorentz transformation is a special kind of coordinate transformation, with $x^{\mu'} = \Lambda^{\mu'}{}_\mu x^\mu$. But (2.19) is much more general, as it encompasses the behavior of vectors under arbitrary changes of coordinates (and therefore bases), not just linear transformations. As usual, we are trying to emphasize a somewhat subtle ontological distinction—in principle, tensor components need not change when we change coordinates, they change when we change the basis in the tangent space, but we have decided to use the coordinates to define our basis. Therefore a change of coordinates induces a change of basis, as indicated in Figure 2.20.

Since a vector at a point can be thought of as a directional derivative operator along a path through that point, it should be clear that a vector *field* defines a map from smooth functions to smooth functions all over the manifold, by taking a

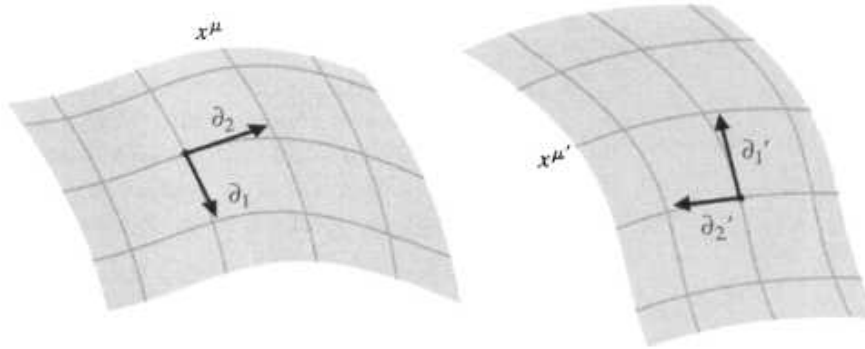


FIGURE 2.20 A change of coordinates $x^\mu \to x^{\mu'}$ induces a change of basis in the tangent space.

derivative at each point. Given two vector fields X and Y, we can therefore define their **commutator** $[X, Y]$ by its action on a function $f(x^\mu)$:

$$[X, Y](f) \equiv X(Y(f)) - Y(X(f)). \tag{2.20}$$

The virtue of the abstract point of view is that, clearly, this operator is independent of coordinates. In fact, the commutator of two vector fields is itself a vector field: if f and g are functions and a and b are real numbers, the commutator is linear,

$$[X, Y](af + bg) = a[X, Y](f) + b[X, Y](g), \tag{2.21}$$

and obeys the Leibniz rule,

$$[X, Y](fg) = f[X, Y](g) + g[X, Y](f). \tag{2.22}$$

Both properties are straightforward to check, which is a useful exercise to do. An equally interesting exercise is to derive an explicit expression for the components of the vector field $[X, Y]^\mu$, which turns out to be

$$\boxed{[X, Y]^\mu = X^\lambda \partial_\lambda Y^\mu - Y^\lambda \partial_\lambda X^\mu.} \tag{2.23}$$

By construction this is a well-defined tensor; but you should be slightly worried by the appearance of the partial derivatives, since partial derivatives of vectors are not well-defined tensors (as we discuss in the next section). Yet another fascinating exercise is to perform explicitly a coordinate transformation on the expression (2.23), to verify that all potentially nontensorial pieces cancel and the result transforms like a vector field. The commutator is a special case of the Lie derivative, discussed in Appendix B; it is sometimes referred to as the **Lie bracket**. Note that since partials commute, the commutator of the vector fields given by the partial derivatives of coordinate functions, $\{\partial_\mu\}$, always vanishes.

2.4 ■ TENSORS AGAIN

Having explored the world of vectors, we continue to retrace the steps we took in flat space, and now consider dual vectors (one-forms). Once again the cotangent space T_p^* can be thought of as the set of linear maps $\omega : T_p \to \mathbf{R}$. The canonical example of a one-form is the gradient of a function f, denoted $\mathrm{d}f$, as in (1.52). Its action on a vector $d/d\lambda$ is exactly the directional derivative of the function:

$$\mathrm{d}f\left(\frac{d}{d\lambda}\right) = \frac{df}{d\lambda}. \tag{2.24}$$

It's tempting to ask, "why shouldn't the function f itself be considered the one-form, and $df/d\lambda$ its action?" The point is that a one-form, like a vector, exists only at the point it is defined, and does not depend on information at other points on M. If you know a function in some neighborhood of a point, you can take its derivative, but not just from knowing its value at the point; the gradient, on the other hand, encodes precisely the information necessary to take the directional derivative along any curve through p, fulfilling its role as a dual vector.

You may have noticed that we defined vectors using structures intrinsic to the manifold (directional derivatives along curves), and used that definition to define one-forms in terms of the dual vector space. This might lead to the impression that vectors are somehow more fundamental; in fact, however, we could just as well have begun with an intrinsic definition of one-forms and used that to define vectors as the dual space. Roughly speaking, the space of one-forms at p is equivalent to the space of all functions that vanish at p and have the same second partial derivatives. In fact, doing it that way is more fundamental, if anything, since we can provide intrinsic definitions of all q-forms (totally antisymmetric tensors with q lower indices), which we will discuss in Section 2.9 (although we will not delve into the specifics of the intrinsic definitions).

Just as the partial derivatives along coordinate axes provide a natural basis for the tangent space, the gradients of the coordinate functions x^μ provide a natural basis for the cotangent space. Recall that in flat space we constructed a basis for T_p^* by demanding that $\hat{\theta}^{(\mu)}(\hat{e}_{(\nu)}) = \delta^\mu_\nu$. Continuing the same philosophy on an arbitrary manifold, we find that (2.24) leads to

$$\mathrm{d}x^\mu(\partial_\nu) = \frac{\partial x^\mu}{\partial x^\nu} = \delta^\mu_\nu. \tag{2.25}$$

Therefore the gradients $\{\mathrm{d}x^\mu\}$ are an appropriate set of basis one-forms; an arbitrary one-form is expanded into components as $\omega = \omega_\mu \, \mathrm{d}x^\mu$.

The transformation properties of basis dual vectors and components follow from what is by now the usual procedure. We obtain, for basis one-forms,

$$\mathrm{d}x^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} \mathrm{d}x^\mu \tag{2.26}$$

and for components,

$$\omega_{\mu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \omega_{\mu}. \tag{2.27}$$

We will usually write the components ω_μ when we speak about a one-form ω.

Just as in flat space, a (k, l) tensor is a multilinear map from a collection of k dual vectors and l vectors to $\mathbf{R}$. Its components in a coordinate basis can be obtained by acting the tensor on basis one-forms and vectors,

$$T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} = T(\mathrm{d}x^{\mu_1}, \ldots, \mathrm{d}x^{\mu_k}, \partial_{\nu_1}, \ldots, \partial_{\nu_l}). \tag{2.28}$$

This is equivalent to the expansion

$$T = T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} \partial_{\mu_1} \otimes \cdots \otimes \partial_{\mu_k} \otimes \mathrm{d}x^{\nu_1} \otimes \cdots \otimes \mathrm{d}x^{\nu_l}. \tag{2.29}$$

The transformation law for general tensors follows the same pattern of replacing the Lorentz transformation matrix used in flat space with a matrix representing more general coordinate transformations:

$$\boxed{T^{\mu'_1 \cdots \mu'_k}{}_{\nu'_1 \cdots \nu'_l} = \frac{\partial x^{\mu'_1}}{\partial x^{\mu_1}} \cdots \frac{\partial x^{\mu'_k}}{\partial x^{\mu_k}} \frac{\partial x^{\nu_1}}{\partial x^{\nu'_1}} \cdots \frac{\partial x^{\nu_l}}{\partial x^{\nu'_l}} T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l}.} \tag{2.30}$$

This tensor transformation law is straightforward to remember, since there really isn't anything else it could be, given the placement of indices.

Actually, however, it is often easier to transform a tensor by taking the identity of basis vectors and one-forms as partial derivatives and gradients at face value, and simply substituting in the coordinate transformation. As an example, consider a symmetric $(0, 2)$ tensor S on a two-dimensional manifold, whose components in a coordinate system $(x^1 = x, x^2 = y)$ are given by

$$S_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & x^2 \end{pmatrix}. \tag{2.31}$$

This can be written equivalently as

$$\begin{aligned} S &= S_{\mu\nu}(\mathrm{d}x^{\mu} \otimes \mathrm{d}x^{\nu}) \\ &= (\mathrm{d}x)^2 + x^2(\mathrm{d}y)^2, \end{aligned} \tag{2.32}$$

where in the last line the tensor product symbols are suppressed for brevity (as will become our custom). Now consider new coordinates

$$\begin{aligned} x' &= \frac{2x}{y} \\ y' &= \frac{y}{2} \end{aligned} \tag{2.33}$$

(valid, for example, when $x > 0$, $y > 0$). These can be immediately inverted to obtain

$$\begin{aligned} x &= x'y' \\ y &= 2y'. \end{aligned} \tag{2.34}$$

Instead of using the tensor transformation law, we can simply use the fact that we know how to take derivatives to express dx^μ in terms of $dx^{\mu'}$. We have

$$\begin{aligned} dx &= y'\,dx' + x'\,dy' \\ dy &= 2\,dy'. \end{aligned} \tag{2.35}$$

We need only plug these expressions directly into (2.32) to obtain (remembering that tensor products don't commute, so $dx'\,dy' \neq dy'\,dx'$):

$$S = (y')^2(dx')^2 + x'y'(dx'\,dy' + dy'\,dx') + [(x')^2 + 4(x'y')^2](dy')^2, \tag{2.36}$$

or

$$S_{\mu'\nu'} = \begin{pmatrix} (y')^2 & x'y' \\ x'y' & (x')^2 + 4(x'y')^2 \end{pmatrix}. \tag{2.37}$$

Notice that it is still symmetric. We did not use the transformation law (2.30) directly, but doing so would have yielded the same result, as you can check.

For the most part the various tensor operations we defined in flat space are unaltered in a more general setting: contraction, symmetrization, and so on. There are three important exceptions: partial derivatives, the metric, and the Levi–Civita tensor. Let's look at the partial derivative first.

Unfortunately, the partial derivative of a tensor is not, in general, a new tensor. The gradient, which is the partial derivative of a scalar, is an honest (0, 1) tensor, as we have seen. But the partial derivative of higher-rank tensors is not tensorial, as we can see by considering the partial derivative of a one-form, $\partial_\mu W_\nu$, and changing to a new coordinate system:

$$\begin{aligned} \frac{\partial}{\partial x^{\mu'}} W_{\nu'} &= \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial}{\partial x^\mu} \left(\frac{\partial x^\nu}{\partial x^{\nu'}} W_\nu \right) \\ &= \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \left(\frac{\partial}{\partial x^\mu} W_\nu \right) + W_\nu \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial}{\partial x^\mu} \frac{\partial x^\nu}{\partial x^{\nu'}}. \end{aligned} \tag{2.38}$$

The second term in the last line should not be there if $\partial_\mu W_\nu$ were to transform as a (0, 2) tensor. As you can see, it arises because the derivative of the transformation matrix does not vanish, as it did for Lorentz transformations in flat space.

Differentiation is obviously an important tool in physics, so we will have to invent new tensorial operations to take the place of the partial derivative. In fact we will invent several: the exterior derivative, the covariant derivative, and the Lie derivative.

2.5 ■ THE METRIC

The metric tensor is such an important object in curved space that it is given a new symbol, $g_{\mu\nu}$ (while $\eta_{\mu\nu}$ is reserved specifically for the Minkowski metric). There are few restrictions on the components of $g_{\mu\nu}$, other than that it be a symmetric (0, 2) tensor. It is usually, though not always, taken to be nondegenerate, meaning that the determinant $g = |g_{\mu\nu}|$ doesn't vanish. This allows us to define the inverse metric $g^{\mu\nu}$ via

$$g^{\mu\nu}g_{\nu\sigma} = g_{\lambda\sigma}g^{\lambda\mu} = \delta^{\mu}_{\sigma}. \tag{2.39}$$

The symmetry of $g_{\mu\nu}$ implies that $g^{\mu\nu}$ is also symmetric. Just as in special relativity, the metric and its inverse may be used to raise and lower indices on tensors. You may be familiar with the notion of a "metric" used in the study of topology, where we also demand that the metric be positive–definite (no negative eigenvalues). The metric we use in general relativity cannot be used to define a topology, but it will have other uses.

It will take some time to fully appreciate the role of the metric in all of its glory, but for purposes of inspiration [following Sachs and Wu (1977)] we can list the various uses to which $g_{\mu\nu}$ will be put: (1) the metric supplies a notion of "past" and "future"; (2) the metric allows the computation of path length and proper time; (3) the metric determines the "shortest distance" between two points, and therefore the motion of test particles; (4) the metric replaces the Newtonian gravitational field ϕ; (5) the metric provides a notion of locally inertial frames and therefore a sense of "no rotation"; (6) the metric determines causality, by defining the speed of light faster than which no signal can travel; (7) the metric replaces the traditional Euclidean three-dimensional dot product of Newtonian mechanics. Obviously these ideas are not all completely independent, but we get some sense of the importance of this tensor.

In our discussion of path lengths in special relativity we (somewhat handwavingly) introduced the line element as $ds^2 = \eta_{\mu\nu}dx^{\mu}dx^{\nu}$, which was used to get the length of a path. Of course now that we know that $\mathrm{d}x^{\mu}$ is really a basis dual vector, it becomes natural to use the terms "metric" and "line element" interchangeably, and write

$$ds^2 = g_{\mu\nu}\,\mathrm{d}x^{\mu}\,\mathrm{d}x^{\nu}. \tag{2.40}$$

To be perfectly consistent we should write this as "g," and sometimes will, but more often than not g is used for the determinant $|g_{\mu\nu}|$. For example, we know that the Euclidean line element in a three-dimensional space with Cartesian coordinates is

$$ds^2 = (\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}z)^2. \tag{2.41}$$

We can now change to any coordinate system we choose. For example, in spherical coordinates we have

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta, \tag{2.42}$$

which leads directly to

$$ds^2 = dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\phi^2. \tag{2.43}$$

Obviously the components of the metric look different than those in Cartesian coordinates, but all of the properties of the space remain unaltered.

Most references are not sufficiently picky to distinguish between "dx," the informal notion of an infinitesimal displacement, and "dx," the rigorous notion of a basis one-form given by the gradient of a coordinate function. (They also tend to neglect the fact that tensor products don't commute, and write expressions like $dxdy + dydx$ as $2dxdy$; it should be clear what is meant from the context.) In fact our notation "ds^2" does not refer to the differential of anything, or the square of anything; it's just conventional shorthand for the metric tensor, a multilinear map from two vectors to the real numbers. Thus, we have a set of equivalent expressions for the inner product of two vectors V^μ and W^ν:

$$g_{\mu\nu}V^\mu W^\nu = g(V, W) = ds^2(V, W). \tag{2.44}$$

Meanwhile, "$(dx)^2$" refers specifically to the honest (0, 2) tensor $dx \otimes dx$.

A good example of a non-Euclidean manifold is the two-sphere, which can be thought of as the locus of points in $\mathbf{R}^3$ at distance 1 from the origin. The metric in the (θ, ϕ) coordinate system can be derived by setting $r = 1$ and $dr = 0$ in (2.43):

$$ds^2 = d\theta^2 + \sin^2\theta\,d\phi^2. \tag{2.45}$$

This is completely consistent with the interpretation of ds as an infinitesimal length, as illustrated in Figure 2.21. Anyone paying attention should at this point be asking, "What in the world does it mean to set $dr = 0$? We know that dr is a well-defined nonvanishing one-form field." As occasionally happens, we are using sloppy language to motivate a step that is actually quite legitimate; see Appendix A for a discussion of how submanifolds inherit metrics from the spaces in which they are embedded.

As we shall see, the metric tensor contains all the information we need to describe the curvature of the manifold (at least in what is called Riemannian geometry; we will get into some of the subtleties in the next chapter). In Minkowski space we can choose coordinates in which the components of the metric are constant; but it should be clear that the existence of curvature is more subtle than having the metric depend on the coordinates, since in the example above we showed how the metric in flat Euclidean space in spherical coordinates is a function of r and θ. Later, we shall see that constancy of the metric components is sufficient for a space to be flat, and in fact there always exists a coordinate system on any

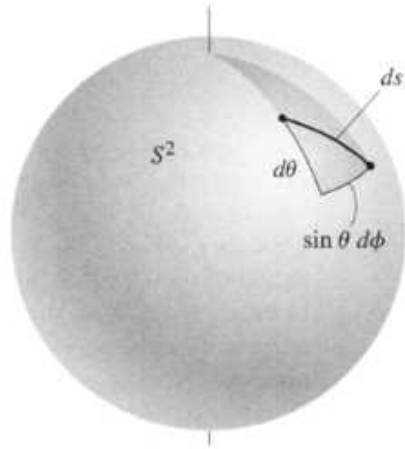


FIGURE 2.21 The line element on a two-dimensional sphere.

flat space in which the metric is constant. But we might not know how to find such a coordinate system, and there are many ways for a space to deviate from flatness; we will therefore want a more precise characterization of the curvature, which will be introduced later.

A useful characterization of the metric is obtained by putting $g_{\mu\nu}$ into its **canonical form**. In this form the metric components become

$$g_{\mu\nu} = \text{diag}\,(-1, -1, \ldots, -1, +1, +1, \ldots, +1, 0, 0, \ldots, 0), \tag{2.46}$$

where "diag" means a diagonal matrix with the given elements. The **signature** of the metric is the number of both positive and negative eigenvalues; we speak of "a metric with signature minus-plus-plus-plus" for Minkowski space, for example. If any of the eigenvalues are zero, the metric is "degenerate," and the inverse metric will not exist; if the metric is continuous and nondegenerate, its signature will be the same at every point. We will always deal with continuous, nondegenerate metrics. If all of the signs are positive, the metric is called **Euclidean** or **Riemannian** (or just positive definite), while if there is a single minus it is called **Lorentzian** or **pseudo-Riemannian**, and any metric with some $+1$'s and some -1's is called indefinite. (So the word Euclidean sometimes means that the space is flat, and sometimes doesn't, but it always means that the canonical form is strictly positive; the terminology is unfortunate but standard.) The spacetimes of interest in general relativity have Lorentzian metrics.

We haven't yet demonstrated that it is always possible to put the metric into canonical form. In fact it is always possible to do so at some point $p \in M$, but in general it will only be possible at that single point, not in any neighborhood of p. Actually we can do slightly better than this; it turns out that at any point p there exists a coordinate system $x^{\hat{\mu}}$ in which $g_{\hat{\mu}\hat{\nu}}$ takes its canonical form and the first derivatives $\partial_{\hat{\sigma}} g_{\hat{\mu}\hat{\nu}}$ all vanish (while the second derivatives $\partial_{\hat{\rho}}\partial_{\hat{\sigma}} g_{\hat{\mu}\hat{\nu}}$ cannot be

made to all vanish):

$$g_{\hat{\mu}\hat{\nu}}(p) = \eta_{\hat{\mu}\hat{\nu}}, \quad \partial_{\hat{\sigma}} g_{\hat{\mu}\hat{\nu}}(p) = 0. \tag{2.47}$$

Such coordinates are known as **locally inertial coordinates**, and the associated basis vectors constitute a **local Lorentz frame**; we often put hats on the indices when we are in these special coordinates. Notice that in locally inertial coordinates the metric at p looks like that of flat space to first order. This is the rigorous notion of the idea that "small enough regions of spacetime look like flat (Minkowski) space." Also, there is no difficulty in simultaneously constructing sets of *basis vectors* at every point in M such that the metric takes its canonical form; the problem is that in general there will not be a *coordinate system* from which this basis can be derived. Bases of this sort are discussed in Appendix J.

We will delay a discussion of how to construct locally inertial coordinates until Chapter 3. It is useful, however, to sketch a proof of their existence for the specific case of a Lorentzian metric in four dimensions. The idea is to consider the transformation law for the metric

$$g_{\hat{\mu}\hat{\nu}} = \frac{\partial x^{\mu}}{\partial x^{\hat{\mu}}} \frac{\partial x^{\nu}}{\partial x^{\hat{\nu}}} g_{\mu\nu}, \tag{2.48}$$

and expand both sides in Taylor series in the sought-after coordinates $x^{\hat{\mu}}$. The expansion of the old coordinates x^{μ} looks like

$$\begin{aligned} x^{\mu} = &\left(\frac{\partial x^{\mu}}{\partial x^{\hat{\mu}}}\right)_p x^{\hat{\mu}} + \frac{1}{2}\left(\frac{\partial^2 x^{\mu}}{\partial x^{\hat{\mu}_1} \partial x^{\hat{\mu}_2}}\right)_p x^{\hat{\mu}_1} x^{\hat{\mu}_2} \\ &+ \frac{1}{6}\left(\frac{\partial^3 x^{\mu}}{\partial x^{\hat{\mu}_1} \partial x^{\hat{\mu}_2} \partial x^{\hat{\mu}_3}}\right)_p x^{\hat{\mu}_1} x^{\hat{\mu}_2} x^{\hat{\mu}_3} + \cdots, \end{aligned} \tag{2.49}$$

with the other expansions proceeding along the same lines. [For simplicity we have set $x^{\mu}(p) = x^{\hat{\mu}}(p) = 0$.] Then, using some extremely schematic notation, the expansion of (2.48) to second order is

$$\begin{aligned} &\left(\hat{g}\right)_p + \left(\hat{\partial}\hat{g}\right)_p \hat{x} + \left(\hat{\partial}\hat{\partial}\hat{g}\right)_p \hat{x}\hat{x} \\ &\quad = \left(\frac{\partial x}{\partial \hat{x}} \frac{\partial x}{\partial \hat{x}} g\right)_p + \left(\frac{\partial x}{\partial \hat{x}} \frac{\partial^2 x}{\partial \hat{x} \partial \hat{x}} g + \frac{\partial x}{\partial \hat{x}} \frac{\partial x}{\partial \hat{x}} \hat{\partial} g\right)_p \hat{x} \\ &\qquad + \left(\frac{\partial x}{\partial \hat{x}} \frac{\partial^3 x}{\partial \hat{x} \partial \hat{x} \partial \hat{x}} g + \frac{\partial^2 x}{\partial \hat{x} \partial \hat{x}} \frac{\partial^2 x}{\partial \hat{x} \partial \hat{x}} g + \frac{\partial x}{\partial \hat{x}} \frac{\partial^2 x}{\partial \hat{x} \partial \hat{x}} \hat{\partial} g + \frac{\partial x}{\partial \hat{x}} \frac{\partial x}{\partial \hat{x}} \hat{\partial}\hat{\partial} g\right)_p \hat{x}\hat{x}. \end{aligned} \tag{2.50}$$

We can set terms of equal order in $\hat{x}$ on each side equal to each other. Therefore, the components $g_{\hat{\mu}\hat{\nu}}(p)$, 10 numbers in all (to describe a symmetric two-index tensor), are determined by the matrix $(\partial x^{\mu}/\partial x^{\hat{\mu}})_p$. This is a 4×4

matrix with no constraints; thus, we are free to choose 16 numbers. Clearly this is enough freedom to put the 10 numbers of $g_{\hat{\mu}\hat{\nu}}(p)$ into canonical form, at least as far as having enough degrees of freedom is concerned. (In fact there are some limitations—if you go through the procedure carefully, you find for example that you cannot change the signature.) The six remaining degrees of freedom can be interpreted as exactly the six parameters of the Lorentz group; we know that these leave the canonical form unchanged. At first order we have the derivatives $\partial_{\hat{\sigma}} g_{\hat{\mu}\hat{\nu}}(p)$, four derivatives of ten components for a total of 40 numbers. But looking at the right-hand side of (2.50) we see that we now have the additional freedom to choose $(\partial^2 x^{\mu}/\partial x^{\hat{\mu}_1}\partial x^{\hat{\mu}_2})_p$. In this set of numbers there are 10 independent choices of the indices $\hat{\mu}_1$ and $\hat{\mu}_2$ (it's symmetric, since partial derivatives commute) and four choices of μ, for a total of 40 degrees of freedom. This is precisely the number of choices we need to determine all of the first derivatives of the metric, which we can therefore set to zero. At second order, however, we are concerned with $\partial_{\hat{\rho}}\partial_{\hat{\sigma}} g_{\hat{\mu}\hat{\nu}}(p)$; this is symmetric in $\hat{\rho}$ and $\hat{\sigma}$ as well as $\hat{\mu}$ and $\hat{\nu}$, for a total of $10 \times 10 = 100$ numbers. Our ability to make additional choices is contained in $(\partial^3 x^{\mu}/\partial x^{\hat{\mu}_1}\partial x^{\hat{\mu}_2}\partial x^{\hat{\mu}_3})_p$. This is symmetric in the three lower indices, which gives 20 possibilities, times four for the upper index gives us 80 degrees of freedom—20 fewer than we require to set the second derivatives of the metric to zero. So in fact we cannot make the second derivatives vanish; the deviation from flatness must therefore be measured by the 20 degrees of freedom representing the second derivatives of the metric tensor field. We will see later how this comes about, when we characterize curvature using the Riemann tensor, which will turn out to have 20 independent components in four dimensions.

Locally inertial coordinates are unbelievably useful. Best of all, their usefulness does not generally require that we actually do the work of constructing such coordinates (although we will give a recipe for doing so in the next chapter), but simply that we know that they do exist. The usual trick is to take a question of physical interest, answer it in the context of locally inertial coordinates, and then express that answer in a coordinate-independent form. Take a very simple example, featuring an observer with four-velocity $U^{\hat{\mu}}$ and a rocket flying past with four-velocity $V^{\hat{\mu}}$. What does the observer measure as the ordinary three-velocity of the rocket? In special relativity the answer is straightforward. Work in inertial coordinates (globally, not just locally) such that the observer is in the rest frame and the rocket is moving along the x-axis. Then the four-velocity of the observer is $U^{\hat{\mu}} = (1, 0, 0, 0)$ and the four-velocity of the rocket is $V^{\hat{\mu}} = (\gamma, v\gamma, 0, 0)$, where v is the three-velocity and $\gamma = 1/\sqrt{1 - v^2}$, so that $v = \sqrt{1 - \gamma^{-2}}$. Since we are in flat spacetime (for the moment), we have

$$\gamma = -\eta_{\hat{\mu}\hat{\nu}} U^{\hat{\mu}} V^{\hat{\nu}} = -U_{\hat{\mu}} V^{\hat{\mu}}, \tag{2.51}$$

since $\eta_{00} = -1$. The flat-spacetime answer would therefore be

$$v = \sqrt{1 - (U_{\hat{\mu}} V^{\hat{\mu}})^{-2}}. \tag{2.52}$$

Now we can go back to curved spacetime, where the metric is no longer flat. But at the point where the measurement is being done, we are free to use locally inertial coordinates, in which case the components of $g_{\hat{\mu}\hat{\nu}}$ are precisely those of $\eta_{\hat{\mu}\hat{\nu}}$. So (2.52) is still true in curved spacetime in this particular coordinate system. But (2.52) is a completely tensorial equation, which doesn't care what coordinate system we are in; therefore it is true in complete generality. This kind of procedure will prove its value over and over.

2.6 ■ AN EXPANDING UNIVERSE

A simple example of a nontrivial Lorentzian geometry is provided by a four-dimensional cosmological spacetime with metric

$$ds^2 = -dt^2 + a^2(t)[dx^2 + dy^2 + dz^2]. \tag{2.53}$$

This describes a universe for which "space at a fixed moment of time" is a flat three-dimensional Euclidean space, which is expanding as a function of time. Worldlines that remain at constant spatial coordinates x^i are said to be comoving; similarly, we denote a region of space that expands along with boundaries defined by fixed spatial coordinates as a "comoving volume." Since the metric describes (distance)^2, the relative distance between comoving points is growing as $a(t)$ in this spacetime; the function a is called the scale factor. This is a special case of a Robertson–Walker metric, one in which spatial slices are geometrically flat; there are other cases for which spatial slices are curved (as we will discuss in Chapter 8). But our interest right now is not in where this metric came from, but in using it as a playground to illustrate some of the ideas we have developed.

Typical solutions for the scale factor are power laws,

$$a(t) = t^q, \qquad 0 < q < 1. \tag{2.54}$$

Actually there are all sorts of solutions, but these are some particularly simple and relevant ones. A matter-dominated flat universe satisfies $q = \frac{2}{3}$, while a radiation-dominated flat universe satisfies $q = \frac{1}{2}$. An obvious feature is that the scale factor goes to zero as $t \to 0$, and along with it the spatial components of the metric. This is a coordinate-dependent statement, and in principle there might be another coordinate system in which everything looks finite; in this case, however, $t = 0$ represents a true singularity of the geometry (the "Big Bang"), and should be excluded from the manifold. The range of the t coordinate is therefore

$$0 < t < \infty. \tag{2.55}$$

Our spacetime comes to an end at $t = 0$.

Light cones in this curved geometry are defined by null paths, those for which $ds^2 = 0$. We can draw a spacetime diagram by considering null paths for which

y and z are held constant; then

$$0 = -\mathrm{d}t^2 + t^{2q}\,\mathrm{d}x^2, \tag{2.56}$$

which implies

$$\frac{dx}{dt} = \pm t^{-q}. \tag{2.57}$$

You might worry that, after all that fuss about $\mathrm{d}x^\mu$ being a basis one-form and not a differential, we have sloppily "divided by $\mathrm{d}t^2$" to go from (2.56) to (2.57). The truth is much more respectable. What we actually did was to take the (0, 2) tensor defined by (2.56), which takes two vectors and returns a real number, and act it on two copies of the vector $V = (dx^\mu/d\lambda)\partial_\mu$, the tangent vector to a curve $x^\mu(\lambda)$. Consider just the $\mathrm{d}t^2$ piece acting on V:

$$\mathrm{d}t^2(V, V) \equiv (\mathrm{d}t \otimes \mathrm{d}t)(V, V) = \mathrm{d}t(V) \cdot \mathrm{d}t(V), \tag{2.58}$$

where the notation $\mathrm{d}t(V)$ refers to a real number that we compute as

$$\begin{aligned} \mathrm{d}t(V) &= \mathrm{d}t\left(\frac{dx^\mu}{d\lambda}\partial_\mu\right) \\ &= \frac{dx^\mu}{d\lambda}\mathrm{d}t\left(\partial_\mu\right) \\ &= \frac{dx^\mu}{d\lambda}\frac{\partial t}{\partial x^\mu} \\ &= \frac{dt}{d\lambda}, \end{aligned} \tag{2.59}$$

where in the third line we have invoked (2.25). Following the same procedure with $\mathrm{d}x^2$, we find that (2.56) implies

$$0 = -\left(\frac{dt}{d\lambda}\right)^2 + t^{2q}\left(\frac{dx}{d\lambda}\right)^2, \tag{2.60}$$

from which (2.57) follows via the one-dimensional chain rule,

$$\frac{dx}{dt} = \frac{dx}{d\lambda}\frac{d\lambda}{dt}. \tag{2.61}$$

The lesson should be clear: expressions such as (2.56) describe well-defined tensors, but manipulation of the basis one-forms as if they were simply "differentials" does get you the right answer. (At least, most of the time; it's a good idea to keep the more formal definitions in mind.)

We can solve (2.57) to obtain

$$t = (1-q)^{1/(1-q)}(\pm x - x_0)^{1/(1-q)}, \tag{2.62}$$

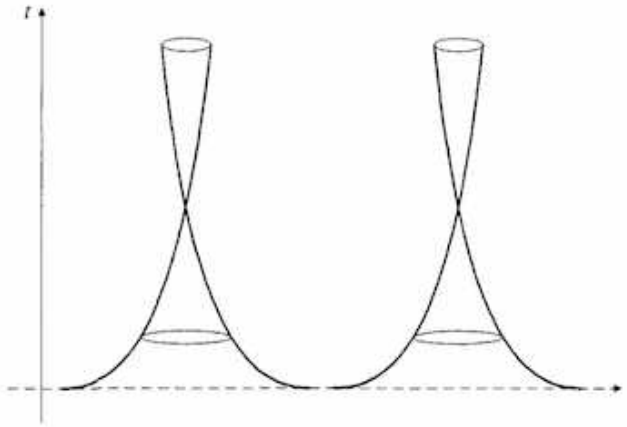


FIGURE 2.22 Spacetime diagram for a flat Robertson–Walker universe with $a(t) \propto t^q$, for $0 < q < 1$. The dashed line at the bottom of the figure represents the singularity at $t = 0$. Since light cones are tangent to the singularity, the pasts of two points may be nonoverlapping.

where x_0 is a constant of integration. These curves define the light cones of our expanding universe, as plotted in Figure 2.22. Since we have assumed $0 < q < 1$, the light cones are tangent to the singularity at $t = 0$. A crucial feature of this geometry is that the light cones of two points need not intersect in the past; this is in contrast to Minkowski space, for which the light cones of any two points always intersect in both the past and future. We say that every event defines an "horizon," outside of which there exist worldlines that can have had no influence on what happens at that event. This is because, since nothing can travel faster than light, each point can only be influenced by events that are either on, or in the interior of, its past light cone (indeed, we refer to the past light cone plus its interior as simply "the past" of an event). Two events outside each others' horizons are said to be "out of causal contact." These notions will be explored more carefully in the next section, as well as in Chapters 4 and 8.

2.7 ■ CAUSALITY

Many physical questions can be cast as an initial-value problem: given the state of a system at some moment in time, what will be the state at some later time? The fact that such questions have definite answers is due to causality, the idea that future events can be understood as consequences of initial conditions plus the laws of physics. Initial-value problems are as common in GR as in Newtonian physics or special relativity; however, the dynamical nature of the spacetime background introduces new ways in which an initial-value formulation could break down. Here we very briefly introduce some of the concepts used in understanding how causality works in GR.

We will look at the problem of evolving matter fields on a fixed background spacetime, rather than the evolution of the metric itself. Our guiding principle will be that no signals can travel faster than the speed of light; therefore information will only flow along timelike or null trajectories (not necessarily geodesics). Since it is sometimes useful to distinguish between purely timelike paths and ones that are merely non-spacelike, we define a **causal curve** to be one which is timelike or null everywhere. Then, given any subset S of a manifold M, we define the **causal future** of S, denoted $J^+(S)$, to be the set of points that can be reached from S by following a future-directed causal curve; the **chronological future** $I^+(S)$ is the set of points that can be reached by following a future-directed timelike curve. Note that a curve of zero length is causal but not chronal; therefore, a point p will always be in its own causal future $J^+(p)$, but not necessarily in its own chronological future $I^+(p)$ (although it could be, as we mention below). The causal past J^- and chronological past I^- are defined analogously.

A subset $S \subset M$ is called **achronal** if no two points in S are connected by a timelike curve; for example, any edgeless spacelike hypersurface in Minkowski spacetime is achronal. Given a closed achronal set S, we define the **future domain of dependence** of S, denoted $D^+(S)$, as the set of all points p such that *every* past-moving inextendible causal curve through p must intersect S. (Inextendible just means that the curve goes on forever, not ending at some finite point; closed means that the complement of the set is an open set.) Elements of S itself are elements of $D^+(S)$. The past domain of dependence $D^-(S)$ is defined by replacing future with past. Generally speaking, some points in M will be in one of the domains of dependence, and some will be outside; we define the boundary of $D^+(S)$ to be the **future Cauchy horizon** $H^+(S)$, and likewise the boundary of $D^-(S)$ to be the past Cauchy horizon $H^-(S)$. You can convince yourself that they are both null surfaces. The domains of dependence and Cauchy horizons are illustrated in Figure 2.23, in which S is taken to be a connected subset of an achronal surface Σ.

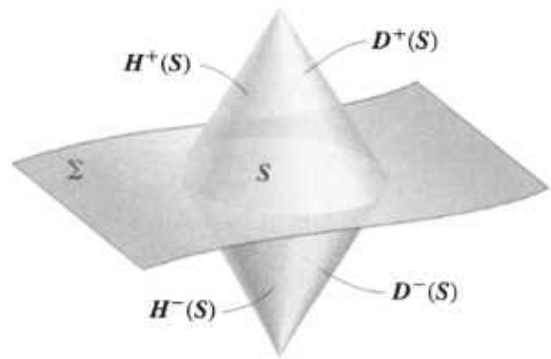


FIGURE 2.23 A connected subset S of a spacelike surface Σ, along with its causal structure. $D^{\pm}(S)$ denotes the future/past domain of dependence of S, and $H^{\pm}(S)$ the future/past Cauchy horizon.

The usefulness of these definitions should be apparent; if nothing moves faster than light, signals cannot propagate outside the light cone of any point p. Therefore, if every curve that remains inside this light cone must intersect S, then information specified on S should be sufficient to predict what the situation is at p; that is, initial data for matter fields given on S can be used to solve for the value of the fields at p. The set of all points for which we can predict what happens by knowing what happens on S is the union $D(S) = D^+(S) \cup D^-(S)$, called simply the domain of dependence. A closed achronal surface Σ is said to be a **Cauchy surface** if the domain of dependence $D(\Sigma)$ is the entire manifold; from information given on a Cauchy surface, we can predict what happens throughout all of spacetime. If a spacetime has a Cauchy surface (which it may not), it is said to be **globally hyperbolic**.

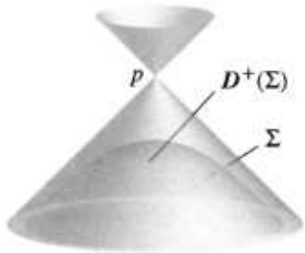


FIGURE 2.24 The surface Σ is everywhere spacelike but lies in the past of the past light cone of the point p; its domain of dependence is not all of the spacetime.

Any set Σ that is closed, achronal, and has no edge, is called a **partial Cauchy surface**. A partial Cauchy surface can fail to be an actual Cauchy surface either through its own fault, or through a fault of the spacetime. One possibility is that we have just chosen a "bad" hypersurface (although it is hard to give a general prescription for when a hypersurface is bad in this sense). Consider Minkowski space, and an edgeless spacelike hypersurface Σ, which remains to the past of the light cone of some point, as in Figure 2.24. In this case Σ is an achronal surface, but it is clear that $D^+(\Sigma)$ ends at the light cone, and we cannot use information on Σ to predict what happens throughout Minkowski space. Of course, there are other surfaces we could have picked for which the domain of dependence would have been the entire manifold, so this doesn't worry us too much.

A somewhat more nontrivial way for a Cauchy horizon to arise is through the appearance of **closed timelike curves**. In Newtonian physics, causality is enforced by the relentless forward march of an absolute notion of time. In special relativity things are even more restrictive; not only must you move forward in time, but the speed of light provides a limit on how swiftly you may move through space (you must stay within your forward light cone). In general relativity it remains true that you must stay within your forward light cone; however, this becomes strictly a local notion, as globally the curvature of spacetime might "tilt" light cones from one place to another. It becomes possible in principle for light cones to be sufficiently distorted that an observer can move on a forward-directed path that is everywhere timelike and yet intersects itself at a point in its "past"—this is a closed timelike curve.

As a simple example, consider a two-dimensional geometry with coordinates $\{t, x\}$, such that points with coordinates (t, x) and $(t, x + 1)$ are identified. The topology is thus $\mathbf{R} \times S^1$. We take the metric to be

$$ds^2 = -\cos(\lambda)dt^2 - \sin(\lambda)[dt\,dx + dx\,dt] + \cos(\lambda)dx^2, \tag{2.63}$$

where

$$\lambda = \cot^{-1} t, \tag{2.64}$$

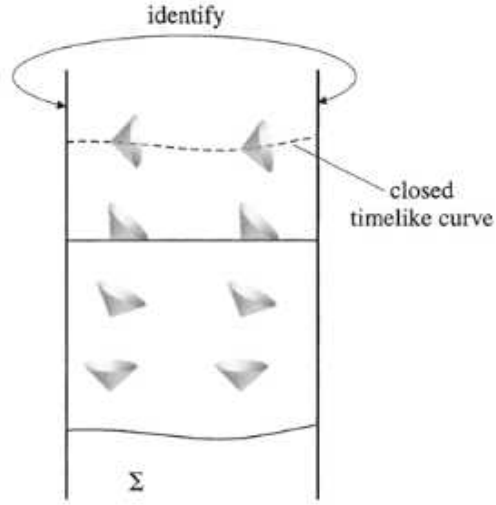


FIGURE 2.25 A cylindrical spacetime with closed timelike curves. The light cones progressively tilt, such that the domain of dependence of the surface Σ fills the lower part of the spacetime, but comes to an end when the closed timelike curves come into existence.

which goes from $\lambda(t = -\infty) = 0$ to $\lambda(t = \infty) = \pi$. This metric doesn't represent any special famous solution to general relativity, it was just cooked up to provide an interesting example of closed timelike curves; but there is a well-known example known as Misner space, with similar properties. In the spacetime defined by (2.63), the light cones progressively tilt as you go forward in time, as shown in Figure 2.25. For $t < 0$, the light cones point forward, and causality is maintained. Once $t > 0$, however, x becomes the timelike coordinate, and it is possible to travel on a timelike trajectory that wraps around the S^1 and comes back to itself; this is a closed timelike curve. If we had specified a surface Σ to this past of this point, then none of the points in the region containing closed timelike curves are in the domain of dependence of Σ, since the closed timelike curves themselves do not intersect Σ. There is thus necessarily a Cauchy horizon at the surface $t = 0$. This is obviously a worse problem than the previous one, since a well-defined initial value problem does not seem to exist in this spacetime.

A final example is provided by the existence of singularities, points that are not in the manifold even though they can be reached by traveling along a geodesic for a finite distance. Typically these occur when the curvature becomes infinite at some point; if this happens, the point can no longer be said to be part of the spacetime. Such an occurrence can lead to the emergence of a Cauchy horizon, as depicted in Figure 2.26—a point p, which is in the future of a singularity, cannot be in the domain of dependence of a hypersurface to the past of the singularity, because there will be curves from p that simply end at the singularity.

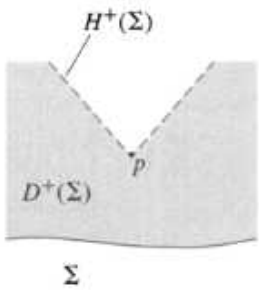


FIGURE 2.26 A singularity at p removes any points in its future from the domain of dependence of a surface Σ in its past.

These obstacles can also arise in the initial value problem for GR, when we try to evolve the metric itself from initial data. However, they are of different degrees of troublesomeness. The possibility of picking a "bad" initial hypersurface does not arise very often, especially since most solutions are found globally (by solving Einstein's equation throughout spacetime). The one situation in which you have to be careful is in numerical solution of Einstein's equation, where a bad choice of hypersurface can lead to numerical difficulties, even if in principle a complete solution exists. Closed timelike curves seem to be something that GR works hard to avoid—there are certainly solutions that contain them, but evolution from generic initial data does not usually produce them. Singularities, on the other hand, are practically unavoidable. The simple fact that the gravitational force is always attractive tends to pull matter together, increasing the curvature, and generally leading to some sort of singularity. Apparently we must learn to live with this, although there is some hope that a well-defined theory of quantum gravity will eliminate (or at least teach us how to deal with) the singularities of classical GR.

2.8 ■ TENSOR DENSITIES

Tensors possess a compelling beauty and simplicity, but there are times when it is useful to consider nontensorial objects. Recall that in Chapter 1 we introduced the completely antisymmetric Levi–Civita symbol, defined as

$$\tilde{\epsilon}_{\mu_1\mu_2\cdots\mu_n} = \begin{cases} +1 & \text{if } \mu_1\mu_2\cdots\mu_n \text{ is an even permutation of } 01\cdots(n-1), \\ -1 & \text{if } \mu_1\mu_2\cdots\mu_n \text{ is an odd permutation of } 01\cdots(n-1), \\ 0 & \text{otherwise.} \end{cases} \tag{2.65}$$

By definition, the Levi–Civita symbol has the components specified above *in any coordinate system* (at least, in any right-handed coordinate system; switching the handedness multiplies the components of $\tilde{\epsilon}_{\mu_1\mu_2\cdots\mu_n}$ by an overall minus sign). This is called a "symbol," of course, because it is not a tensor; it is defined not to change under coordinate transformations. We were only able to treat it as a tensor in inertial coordinates in flat spacetime, since Lorentz transformations would have left the components invariant anyway. Its behavior can be related to that of an ordinary tensor by first noting that, given any $n \times n$ matrix $M^{\mu}{}_{\mu'}$, the determinant $|M|$ obeys

$$\tilde{\epsilon}_{\mu'_1\mu'_2\cdots\mu'_n}|M| = \tilde{\epsilon}_{\mu_1\mu_2\cdots\mu_n} M^{\mu_1}{}_{\mu'_1} M^{\mu_2}{}_{\mu'_2} \cdots M^{\mu_n}{}_{\mu'_n}. \tag{2.66}$$

This is just a streamlined expression for the determinant of any matrix, completely equivalent to the usual formula in terms of matrices of cofactors. (You can check it for yourself for 2×2 or 3×3 matrices.) It follows that, setting $M^{\mu}{}_{\mu'} = \partial x^{\mu}/\partial x^{\mu'}$, we have

$$\tilde{\epsilon}_{\mu_1'\mu_2'\cdots\mu_n'} = \left|\frac{\partial x^{\mu'}}{\partial x^{\mu}}\right| \tilde{\epsilon}_{\mu_1\mu_2\cdots\mu_n} \frac{\partial x^{\mu_1}}{\partial x^{\mu_1'}} \frac{\partial x^{\mu_2}}{\partial x^{\mu_2'}} \cdots \frac{\partial x^{\mu_n}}{\partial x^{\mu_n'}}, \tag{2.67}$$

where we have also used the facts that the matrix $\partial x^{\mu'}/\partial x^{\mu}$ is the inverse of $\partial x^{\mu}/\partial x^{\mu'}$, and that the determinant of an inverse matrix is the inverse of the determinant, $|M^{-1}| = |M|^{-1}$. So the Levi–Civita symbol transforms in a way close to the tensor transformation law, except for the determinant out front. Objects transforming in this way are known as **tensor densities**. Another example is given by the determinant of the metric, $g = |g_{\mu\nu}|$. It's easy to check, by taking the determinant of both sides of (2.48), that under a coordinate transformation we get

$$g(x^{\mu'}) = \left|\frac{\partial x^{\mu'}}{\partial x^{\mu}}\right|^{-2} g(x^{\mu}). \tag{2.68}$$

Therefore g is also not a tensor; it transforms in a way similar to the Levi–Civita symbol, except that the Jacobian is raised to the -2 power. The power to which the Jacobian is raised is known as the **weight** of the tensor density; the Levi–Civita symbol is a density of weight 1, while g is a (scalar) density of weight -2.

However, we don't like tensor densities as much as we like tensors. There is a simple way to convert a density into an honest tensor—multiply by $|g|^{w/2}$, where w is the weight of the density (the absolute value signs are there because $g < 0$ for Lorentzian metrics). The result will transform according to the tensor transformation law. Therefore, for example, we can define the **Levi–Civita tensor** as

$$\boxed{\epsilon_{\mu_1\mu_2\cdots\mu_n} = \sqrt{|g|}\,\tilde{\epsilon}_{\mu_1\mu_2\cdots\mu_n}.} \tag{2.69}$$

Since this is a real tensor, we can raise indices and so on. Sometimes people define a version of the Levi–Civita symbol with upper indices, $\tilde{\epsilon}^{\mu_1\mu_2\cdots\mu_n}$, whose components are numerically equal to $\mathrm{sgn}(g)\tilde{\epsilon}_{\mu_1\mu_2\cdots\mu_n}$, where $\mathrm{sgn}(g)$ is the sign of the metric determinant. This turns out to be a density of weight -1, and is related to the tensor with upper indices (obtained by using $g^{\mu\nu}$ to raise indices on $\epsilon_{\mu_1\mu_2\cdots\mu_n}$) by

$$\epsilon^{\mu_1\mu_2\cdots\mu_n} = \frac{1}{\sqrt{|g|}}\,\tilde{\epsilon}^{\mu_1\mu_2\cdots\mu_n}. \tag{2.70}$$

Something you often end up doing is contracting p indices on $\epsilon^{\mu_1\mu_2\cdots\mu_n}$ with $\epsilon_{\mu_1\mu_2\cdots\mu_n}$; the result can be expressed in terms of an antisymmetrized product of Kronecker deltas as

$$\epsilon^{\mu_1\mu_2\cdots\mu_p\alpha_1\cdots\alpha_{n-p}}\epsilon_{\mu_1\mu_2\cdots\mu_p\beta_1\cdots\beta_{n-p}} = (-1)^s p!(n-p)!\delta^{[\alpha_1}_{\beta_1}\cdots\delta^{\alpha_{n-p}]}_{\beta_{n-p}}, \tag{2.71}$$

where s is the number of negative eigenvalues of the metric (for Lorentzian signature with our conventions, $s = 1$). The most common example is $p = n - 1$,

for which we have

$$\epsilon^{\mu_1\mu_2\cdots\mu_{n-1}\alpha}\epsilon_{\mu_1\mu_2\cdots\mu_{n-1}\beta} = (-1)^s(n-1)!\,\delta^\alpha_\beta. \tag{2.72}$$

2.9 ■ DIFFERENTIAL FORMS

Let us now introduce a special class of tensors, known as **differential forms** (or just forms). A differential p-form is simply a $(0, p)$ tensor that is completely antisymmetric. Thus, scalars are automatically 0-forms, and dual vectors are automatically one-forms (thus explaining this terminology from before). We also have the 4-form $\epsilon_{\mu\nu\rho\sigma}$. The space of all p-forms is denoted Λ^p, and the space of all p-form fields over a manifold M is denoted $\Lambda^p(M)$. A semi-straightforward exercise in combinatorics reveals that the number of linearly independent p-forms on an n-dimensional vector space is $n!/(p!(n-p)!)$. So at a point on a four-dimensional spacetime there is one linearly independent 0-form, four 1-forms, six 2-forms, four 3-forms, and one 4-form. There are no p-forms for $p > n$, since all of the components will automatically be zero by antisymmetry.

Why should we care about differential forms? This question is hard to answer without some more work, but the basic idea is that forms can be both differentiated and integrated, without the help of any additional geometric structure. We will glance briefly at both of these operations.

Given a p-form A and a q-form B, we can form a $(p+q)$-form known as the **wedge product** $A \wedge B$ by taking the antisymmetrized tensor product:

$$(A \wedge B)_{\mu_1\cdots\mu_{p+q}} = \frac{(p+q)!}{p!\,q!}A_{[\mu_1\cdots\mu_p}B_{\mu_{p+1}\cdots\mu_{p+q}]}. \tag{2.73}$$

Thus, for example, the wedge product of two 1-forms is

$$(A \wedge B)_{\mu\nu} = 2A_{[\mu}B_{\nu]} = A_\mu B_\nu - A_\nu B_\mu. \tag{2.74}$$

Note that

$$A \wedge B = (-1)^{pq}B \wedge A, \tag{2.75}$$

so you can alter the order of a wedge product if you are careful with signs. We are free to suppress indices when using forms, since we know that all of the indices are downstairs and the tensors are completely antisymmetric.

The **exterior derivative** d allows us to differentiate p-form fields to obtain $(p+1)$-form fields. It is defined as an appropriately normalized, antisymmetrized partial derivative:

$$(\mathrm{d}A)_{\mu_1\cdots\mu_{p+1}} = (p+1)\partial_{[\mu_1}A_{\mu_2\cdots\mu_{p+1}]}. \tag{2.76}$$

The simplest example is the gradient, which is the exterior derivative of a 0-form:

$$(\mathrm{d}\phi)_\mu = \partial_\mu\phi. \tag{2.77}$$

Exterior derivatives obey a modified version of the Leibniz rule when applied to the product of a p-form ω and a q-form η:

$$\mathrm{d}(\omega \wedge \eta) = (\mathrm{d}\omega) \wedge \eta + (-1)^p \omega \wedge (\mathrm{d}\eta). \tag{2.78}$$

You are encouraged to prove this yourself.

The reason why the exterior derivative deserves special attention is that *it is a tensor*, even in curved spacetimes, unlike its cousin the partial derivative. For $p = 1$ we can see this from the transformation law for the partial derivative of a one form, (2.38); the offending nontensorial term can be written

$$W_\nu \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial}{\partial x^\mu} \frac{\partial x^\nu}{\partial x^{\nu'}} = W_\nu \frac{\partial^2 x^\nu}{\partial x^{\mu'} \partial x^{\nu'}}. \tag{2.79}$$

This expression is symmetric in μ' and ν', since partial derivatives commute. But the exterior derivative is defined to be the antisymmetrized partial derivative, so this term vanishes (the antisymmetric part of a symmetric expression is zero). We are then left with the correct tensor transformation law; extension to arbitrary p is straightforward. So the exterior derivative is a legitimate tensor operator; it is not, however, an adequate substitute for the partial derivative, since it is only defined on forms. In the next chapter we will define a covariant derivative, which is closer to what we might think of as the extension of the partial derivative to arbitrary manifolds.

Another interesting fact about exterior differentiation is that, for any form A,

$$\mathrm{d}(\mathrm{d}A) = 0, \tag{2.80}$$

which is often written $\mathrm{d}^2 = 0$. This identity is a consequence of the definition of d and the fact that partial derivatives commute, $\partial_\alpha \partial_\beta = \partial_\beta \partial_\alpha$ (acting on anything). This leads us to the following mathematical aside, just for fun. We define a p-form A to be **closed** if $\mathrm{d}A = 0$, and **exact** if $A = \mathrm{d}B$ for some $(p-1)$-form B. Obviously, all exact forms are closed, but the converse is not necessarily true. On a manifold M, closed p-forms comprise a vector space $Z^p(M)$, and exact forms comprise a vector space $B^p(M)$. Define a new vector space, consisting of elements called cohomology classes, as the closed forms modulo the exact forms:

$$H^p(M) = \frac{Z^p(M)}{B^p(M)}. \tag{2.81}$$

That is, two closed forms [elements of $Z^p(M)$] define the same cohomology class [elements of $H^p(M)$] if they differ by an exact form [an element of $B^p(M)$]. Miraculously, the dimensionality of the cohomology spaces $H^p(M)$ depends only on the topology of the manifold M. Minkowski space is topologically equivalent to $\mathbf{R}^4$, which is uninteresting, so that all of the $H^p(M)$ vanish for $p > 0$; for $p = 0$ we have $H^0(M) = \mathbf{R}$. Therefore in Minkowski space all closed forms are exact except for zero-forms; zero-forms can't be exact since there are no -1-

forms for them to be the exterior derivative of. It is striking that information about the topology can be extracted in this way, which essentially involves the solutions to differential equations.

The final operation on differential forms we will introduce is **Hodge duality**. We define the *Hodge star operator* on an n-dimensional manifold as a map from p-forms to $(n-p)$-forms,

$$(*A)_{\mu_1\cdots\mu_{n-p}} = \frac{1}{p!}\epsilon^{\nu_1\cdots\nu_p}{}_{\mu_1\cdots\mu_{n-p}}A_{\nu_1\cdots\nu_p}, \tag{2.82}$$

mapping A to "A dual." Unlike our other operations on forms, the Hodge dual does depend on the metric of the manifold [which should be obvious, since we had to raise some indices on the Levi–Civita tensor in order to define (2.82)]. Applying the Hodge star twice returns either plus or minus the original form:

$$**A = (-1)^{s+p(n-p)}A, \tag{2.83}$$

where s is the number of minus signs in the eigenvalues of the metric.

Two facts on the Hodge dual: First, "duality" in the sense of Hodge is distinct from the relationship between vectors and dual vectors. The idea of "duality" is that of a transformation from one space to another with the property that doing the transformation twice gets you back to the original space. It should be clear that this holds true for both the duality between vectors and one-forms, and the Hodge duality between p-forms and $(n-p)$-forms. A requirement of dualities between vector spaces is that the original and transformed spaces have the same dimensionality; this is true of the spaces of p- and $(n-p)$-forms.

The second fact concerns differential forms in three-dimensional Euclidean space. The Hodge dual of the wedge product of two 1-forms gives another 1-form:

$$*(U\wedge V)_i = \epsilon_i{}^{jk}U_jV_k. \tag{2.84}$$

(All of the prefactors cancel.) Since 1-forms in Euclidean space are just like vectors, we have a map from two vectors to a single vector. You should convince yourself that this is just the conventional cross product, and that the appearance of the Levi–Civita tensor explains why the cross product changes sign under parity (interchange of two coordinates, or equivalently basis vectors). This is why the cross product only exists in three dimensions—because only in three dimensions do we have an interesting map from two dual vectors to a third dual vector.

Electrodynamics provides an especially compelling example of the use of differential forms. From the definition of the exterior derivative, it is clear that equation (1.97) can be concisely expressed as closure of the two-form $F_{\mu\nu}$:

$$\mathrm{d}F = 0. \tag{2.85}$$

Does this mean that F is also exact? Yes; as we've noted, Minkowski space is topologically trivial, so all closed forms are exact. There must therefore be a one-

form A_μ such that

$$F = \mathrm{d}A. \tag{2.86}$$

This one-form is the familiar **vector potential** of electromagnetism, with the 0 component given by the scalar potential, $A_0 = \Phi$, as we discussed in Chapter 1. Gauge invariance is expressed by the observation that the theory is invariant under $A \to A + \mathrm{d}\lambda$ for some scalar (zero-form) λ, and this is also immediate from the relation (2.86). The other one of Maxwell's equations, (1.96), can be expressed as an equation between three-forms:

$$\mathrm{d}(*F) = *J, \tag{2.87}$$

where the current one-form J is just the current four-vector with index lowered. Filling in the details is left for you, as good practice converting from differential-form notation to ordinary index notation.

Hodge duality is intimately related to a fascinating feature of certain field theories: duality between strong and weak coupling. It's hard not to notice that the equations (2.85) and (2.87) look very similar. Indeed, if we set $J_\mu = 0$, the equations are invariant under the "duality transformations"

$$\begin{aligned} F &\to *F, \\ *F &\to -F. \end{aligned} \tag{2.88}$$

We therefore say that the vacuum Maxwell's equations are duality invariant, while the invariance is spoiled in the presence of charges. We might imagine that magnetic as well as electric monopoles existed in nature; then we could add a magnetic current term $*J_M$ to the right-hand side of (2.85), and the equations would be invariant under duality transformations plus the additional replacement $J \leftrightarrow J_M$. (Of course a nonzero right-hand side to (2.85) is inconsistent with $F = \mathrm{d}A$, so this idea only works if A_μ is not a fundamental variable.) Dirac considered the idea of magnetic monopoles and showed that a necessary condition for their existence is that the fundamental monopole charge be inversely proportional to the fundamental electric charge. Now, the fundamental electric charge is a small number; electrodynamics is *weakly coupled*, which is why perturbation theory is so remarkably successful in quantum electrodynamics (QED). But Dirac's condition on magnetic charges implies that a duality transformation takes a theory of weakly coupled electric charges to a theory of strongly coupled magnetic monopoles (and vice-versa). Unfortunately monopoles don't fit easily into ordinary electromagnetism, so these ideas aren't directly applicable; but some sort of duality symmetry may exist in certain theories (such as supersymmetric nonabelian gauge theories). If it did, we would have the opportunity to analyze a theory that looked strongly coupled (and therefore hard to solve) by looking at the weakly coupled dual version; this is exactly what happens in certain theories. The hope is that these techniques will allow us to explore various phenomena that we know exist in strongly coupled quantum field theories, such as confinement of quarks in hadrons.

2.10 ■ INTEGRATION

An important appearance of both tensor densities and differential forms is in integration on manifolds. You have probably been exposed to the fact that in ordinary calculus on $\mathbf{R}^n$ the volume element $d^n x$ picks up a factor of the Jacobian under change of coordinates:

$$d^n x' = \left| \frac{\partial x^{\mu'}}{\partial x^{\mu}} \right| d^n x. \tag{2.89}$$

There is actually a beautiful explanation of this formula from the point of view of differential forms, which arises from the following fact: *on an n-dimensional manifold M, the integrand is properly understood as an n-form.* In other words, an integral over an n-dimensional region $\Sigma \subset M$ is a map from an n-form field ω to the real numbers:

$$\int_{\Sigma} : \omega \rightarrow \mathbf{R}. \tag{2.90}$$

Such a statement may seem strange, but it certainly looks familiar in the context of line integrals. In one dimension any one-form can be written $\omega = \omega(x)\mathrm{d}x$, where the first ω is a one-form and $\omega(x)$ denotes the (single) component function. And indeed, we write integrals in one dimension as $\int \omega(x)\mathrm{d}x$; you may be used to thinking of the symbol $\mathrm{d}x$ as an infinitesimal distance, but it is more properly a differential form.

To make this more clear, consider more than one dimension. If we are claiming that the integrand is an n-form, we need to explain in what sense it is antisymmetric, and for that matter why it is a $(0, n)$ tensor (a linear map from a set of n vectors to $\mathbf{R}$) at all. We all agree that integrals can be written as $\int f(x)\, d\mu$, where $f(x)$ is a scalar function on the manifold and $d\mu$ is the volume element, or measure. The role of the volume element is to assign to every (infinitesimal) region an (infinitesimal) real number, namely the volume of that region. A nice feature of infinitesimal regions (as opposed to ones of finite size) is that they can be taken to be rectangular parallelepipeds—in the presence of curvature we have no clear sense of what a "rectangular parallelepiped" is supposed to mean, but the effects of curvature can be neglected when we work in infinitesimal regions. Clearly we are not being rigorous here, but our present purpose is exclusively motivational.

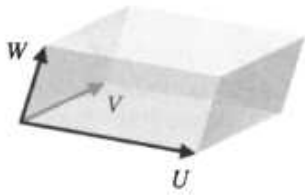


FIGURE 2.27 An infinitesimal n-dimensional region, represented as a parallelepiped, is defined by an ordered set of n vectors, shown here as U, V, and W.

As shown in Figure 2.27 (in which we take our manifold to be three-dimensional for purposes of illustration), a parallelepiped is specified by n vectors that define its edges. Our volume element, then, should be a map from n vectors to the real numbers: $d\mu(U, V, W) \in \mathbf{R}$. (Actually it should be a map from infinitesimal vectors to infinitesimal numbers, but such a map also will take finite vectors to finite numbers.) It's also clear that it should be linearly scalable by real numbers; if we change the length of any of the defining vectors, the volume changes accordingly: $d\mu(aU, bV, cW) = abc\, d\mu(U, V, W)$. Linearity with respect to adding vectors is not so obvious, but you can convince yourself by drawing pictures.

Therefore our volume element is an honest $(0, n)$ tensor. Why antisymmetric? Because we are defining an oriented element; if two of the vectors are interchanged we should get a volume of the same magnitude but opposite sign. (If this is not obvious, you should at least convince yourself that the volume should vanish when two vectors are collinear.) Thus, volume elements in n dimensions are in a very real sense n-forms.

To actually do calculations, we need to make these ideas more concrete, which turns out to be straightforward. The essential insight is to identify the naive volume element $d^n x$ as an antisymmetric tensor density constructed with wedge products:

$$d^n x = \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{n-1}. \tag{2.91}$$

The expression on the right-hand side can be misleading, because it looks like a tensor (an n-form, actually) but is really a density. Certainly if we have two functions f and g on M, then $\mathrm{d}f$ and $\mathrm{d}g$ are one-forms, and $\mathrm{d}f \wedge \mathrm{d}g$ is a two-form. But the functions appearing in (2.91) are the coordinate functions themselves, so when we change coordinates we replace the one-forms $\mathrm{d}x^\mu$ with a new set $\mathrm{d}x^{\mu'}$. You see the funny business—ordinarily a coordinate transformation changes components, but not one-forms themselves. The right-hand side of (2.91) is a coordinate-dependent object (a tensor density, to be precise) which, in the x^μ coordinate system, acts like $\mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{n-1}$. Let's see this in action. First notice that the definition of the wedge product allows us to write

$$\mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{n-1} = \frac{1}{n!}\tilde{\epsilon}_{\mu_1 \cdots \mu_n}\, \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_n}, \tag{2.92}$$

since both the wedge product and the Levi–Civita symbol are completely antisymmetric. (The factor of $1/n!$ takes care of the overcounting introduced by summing over permutations of the indices.) Under a coordinate transformation $\tilde{\epsilon}_{\mu_1 \cdots \mu_n}$ stays the same, while the one-forms change according to (2.26), leading to

$$\begin{aligned}\tilde{\epsilon}_{\mu_1 \cdots \mu_n}\, \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_n} &= \tilde{\epsilon}_{\mu_1 \cdots \mu_n} \frac{\partial x^{\mu_1}}{\partial x^{\mu_1'}} \cdots \frac{\partial x^{\mu_n}}{\partial x^{\mu_n'}}\, \mathrm{d}x^{\mu_1'} \wedge \cdots \wedge \mathrm{d}x^{\mu_n'} \\ &= \left|\frac{\partial x^\mu}{\partial x^{\mu'}}\right| \tilde{\epsilon}_{\mu_1' \cdots \mu_n'}\, \mathrm{d}x^{\mu_1'} \wedge \cdots \wedge \mathrm{d}x^{\mu_n'}. \end{aligned} \tag{2.93}$$

Multiplying by the Jacobian on both sides and using (2.91) and (2.92) recovers (2.89).

It is clear that the naive volume element $d^n x$ transforms as a density, not a tensor, but it is straightforward to construct an invariant volume element by multiplying by $\sqrt{|g|}$:

$$\sqrt{|g'|}\, \mathrm{d}x^{0'} \wedge \cdots \wedge \mathrm{d}x^{(n-1)'} = \sqrt{|g|}\, \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{n-1}, \tag{2.94}$$

which is of course just $(n!)^{-1}\epsilon_{\mu_1 \cdots \mu_n}\, \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_n}$. In the interest of simplicity we will usually write the volume element as $\sqrt{|g|}\, d^n x$, rather than as the explicit

wedge product:

$$\sqrt{|g|}\, d^n x \equiv \sqrt{|g|}\, \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{n-1}\,; \tag{2.95}$$

it will be enough to keep in mind that it's supposed to be an n-form. In fact, the volume element is no more or less than the Levi–Civita tensor $\epsilon_{\mu_1 \cdots \mu_n}$; restoring the explicit basis one-forms, we see

$$\begin{aligned}
\epsilon &\equiv \epsilon_{\mu_1 \cdots \mu_n} \mathrm{d}x^{\mu_1} \otimes \cdots \otimes \mathrm{d}x^{\mu_n} \\
&= \frac{1}{n!} \epsilon_{\mu_1 \cdots \mu_n} \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_n} \\
&= \frac{1}{n!} \sqrt{|g|}\, \tilde{\epsilon}_{\mu_1 \cdots \mu_n} \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_n} \\
&= \sqrt{|g|}\, \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{n-1} \\
&\equiv \sqrt{|g|}\, d^n x.
\end{aligned} \tag{2.96}$$

Notice that the combinatorial factors introduced by the epsilon tensor precisely cancel those from switching from tensor products to wedge products, which is only allowed because the epsilon tensor automatically antisymmetrizes.

The punch line, then, is simple: the integral I of a scalar function ϕ over an n-manifold is written as

$$\boxed{I = \int \phi(x) \sqrt{|g|}\, d^n x.} \tag{2.97}$$

Given explicit forms for $\phi(x)$ and $\sqrt{|g|}$, such an integral can be directly evaluated by the usual methods of multivariable calculus. The metric determinant serves to automatically take care of the correct transformation properties. You will sometimes see the more abstract notation

$$I = \int \phi(x) \epsilon\,; \tag{2.98}$$

given (2.96), these two versions convey the same content.

2.11 ■ EXERCISES

1. Just because a manifold is topologically nontrivial doesn't necessarily mean it can't be covered with a single chart. In contrast to the circle S^1, show that the infinite cylinder $\mathbf{R} \times S^1$ can be covered with just one chart, by explicitly constructing the map.

2. By clever choice of coordinate charts, can we make $\mathbf{R}^2$ look like a one-dimensional manifold? Can we make $\mathbf{R}^1$ look like a two-dimensional manifold? If so, explicitly construct an appropriate atlas, and if not, explain why not. The point of this problem

is to provoke you to think deeply about what a manifold is; it can't be answered rigorously without going into more details about topological spaces. In particular, you might have to forget that you already know a definition of "open set" in the original $\mathbf{R}^2$ or $\mathbf{R}^1$, and define them as being appropriately inherited from the $\mathbf{R}^1$ or $\mathbf{R}^2$ to which they are being mapped.

3. Show that the two-dimensional torus T^2 is a manifold, by explicitly constructing an appropriate atlas. (Not a maximal one, obviously.)

4. Verify the claims made about the commutator of two vector fields at the end of Section 2.3 (linearity, Leibniz, component formula, transformation as a vector field).

5. Give an example of two linearly independent, nowhere-vanishing vector fields in $\mathbf{R}^2$ whose commutator does not vanish. Notice that these fields provide a basis for the tangent space at each point, but it cannot be a coordinate basis since the commutator doesn't vanish.

6. Consider $\mathbf{R}^3$ as a manifold with the flat Euclidean metric, and coordinates $\{x, y, z\}$. Introduce spherical polar coordinates $\{r, \theta, \phi\}$ related to $\{x, y, z\}$ by

$$\begin{aligned} x &= r \sin\theta \cos\phi \\ y &= r \sin\theta \sin\phi \\ z &= r \cos\theta, \end{aligned} \tag{2.99}$$

so that the metric takes the form

$$ds^2 = \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 + r^2 \sin^2\theta \mathrm{d}\phi^2. \tag{2.100}$$

(a) A particle moves along a parameterized curve given by

$$x(\lambda) = \cos\lambda, \quad y(\lambda) = \sin\lambda, \quad z(\lambda) = \lambda. \tag{2.101}$$

Express the path of the curve in the $\{r, \theta, \phi\}$ system.

(b) Calculate the components of the tangent vector to the curve in both the Cartesian and spherical polar coordinate systems.

7. Prolate spheroidal coordinates can be used to simplify the Kepler problem in celestial mechanics. They are related to the usual cartesian coordinates (x, y, z) of Euclidean three-space by

$$\begin{aligned} x &= \sinh\chi \ \sin\theta \ \cos\phi, \\ y &= \sinh\chi \ \sin\theta \ \sin\phi, \\ z &= \cosh\chi \ \cos\theta. \end{aligned}$$

Restrict your attention to the plane $y = 0$ and answer the following questions.

(a) What is the coordinate transformation matrix $\partial x^\mu / \partial x^{\nu'}$ relating (x, z) to (χ, θ)?

(b) What does the line element ds^2 look like in prolate spheroidal coordinates?

8. Verify (2.78): for the exterior derivative of a product of a p-form ω and a q-form η, we have

$$\mathrm{d}(\omega \wedge \eta) = (\mathrm{d}\omega) \wedge \eta + (-1)^p \omega \wedge (\mathrm{d}\eta). \tag{2.102}$$

9. In Minkowski space, suppose $*F = q \sin\theta \, d\theta \wedge d\phi$.

(a) Evaluate $d * F = *J$.

(b) What is the two-form F equal to?

(c) What are the electric and magnetic fields equal to for this solution?

(d) Evaluate $\int_V d*F$, where V is a ball of radius R in Euclidean three space at a fixed moment of time.

10. Consider Maxwell's equations, $dF = 0$, $d*F = *J$, in 2-dimensional spacetime. Explain why one of the two sets of equations can be discarded. Show that the electromagnetic field can be expressed in terms of a scalar field. Write out the field equations for this scalar field in component form.

11. There are a lot of motivational words attached here to what is a very simple problem; don't get too distracted. In ordinary electromagnetism with point particles, the part of the action which represents the coupling of the gauge-potential one-form $A^{(1)}$ to a charged particle can be written $S = \int_\gamma A^{(1)}$, where γ is the particle worldline. (The superscript on $A^{(1)}$ is just to remind you that it is a one-form.) For this problem you will consider a theory related to ordinary electromagnetism, but this time in 11 spacetime dimensions, with a three-form gauge potential $A^{(3)}$ and four-form field strength $F^{(4)} = dA^{(3)}$. Note that the field strength is invariant under a gauge transformation $A^{(3)} \to A^{(3)} + d\lambda^{(2)}$ for any two-form $\lambda^{(2)}$.

(a) What would be the number of spatial dimensions of an object to which this gauge field would naturally couple (for example, ordinary E+M couples to zero-dimensional objects—point particles)?

(b) The electric charge of an ordinary electron is given by the integral of the dual of the two-form gauge field strength over a two-sphere surrounding the particle. How would you define the "charge" of the object to which $A^{(3)}$ couples? Argue that it is conserved if $d * F^{(4)} = 0$.

(c) Imagine there is a "dual gauge potential" $\widetilde{A}$ that satisfies $d(\widetilde{A}) = *F^{(4)}$. To what dimensionality object does it naturally couple?

(d) The action for the gauge field itself (as opposed to its coupling to other things) will be an integral over the entire 11-dimensional spacetime. What are the terms that would be allowed in such an action that are invariant under "local" gauge transformations, for instance, gauge transformations specified by a two-form $\lambda^{(2)}$ that vanishes at infinity? Restrict yourself to terms of first, second, or third order in $A^{(3)}$ and its first derivatives (no second derivatives, no higher-order terms). You may use the exterior derivative, wedge product, and Hodge dual, but not any explicit appearance of the metric.

More background: "Supersymmetry" is a hypothetical symmetry relating bosons (particles with integral spin) and fermions (particles with spin $\frac{1}{2}$, $\frac{3}{2}$, etc.). An interesting feature is that supersymmetric theories are only well-defined in 11 dimensions or less—in larger numbers of dimensions, supersymmetry would require the existence of particles with spins greater than 2, which cannot be consistently quantized. Eleven-dimensional supersymmetry is a unique theory, which naturally includes a three-form gauge potential (not to mention gravity). Recent work has shown that it also includes the various higher-dimensional objects alluded to in this problem (although we've cut some corners here). This theory turns out to be a well-defined limit of something called M-theory, which has as other limits various 10-dimensional superstring theories.

CHAPTER

3 Curvature

3.1 ■ OVERVIEW

We all know what curvature means, at least informally, and in the first two chapters of this book we have felt free to refer on occasion to the concept of curvature without giving it a careful definition. Clearly curvature depends somehow on the metric, which defines the geometry of our manifold; but it is not immediately clear how we should attribute curvature to any given metric (since, as we have seen, even the metric of a flat space can look arbitrarily complicated in a sufficiently extravagant coordinate system). As is often the case in mathematics, we require quite a bit of care to formalize our intuition about a concept into a usable mathematical structure; formalizing what we think of as "curvature" is the subject of this chapter.

The techniques we are about to develop are absolutely crucial to the subject; it is safe to say that there is a higher density of useful formulas per page in this chapter than in any of the others. Let's quickly summarize the most important ones, to provide a roadmap for the formalism to come.

All the ways in which curvature manifests itself rely on something called a "connection," which gives us a way of relating vectors in the tangent spaces of nearby points. There is a unique connection that we can construct from the metric, and it is encapsulated in an object called the *Christoffel symbol*, given by

$$\Gamma^{\lambda}_{\mu\nu} = \tfrac{1}{2} g^{\lambda\sigma} (\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}). \tag{3.1}$$

The notation makes $\Gamma^{\lambda}_{\mu\nu}$ look like a tensor, but in fact it is not; this is why we call it an "object" or "symbol." The fundamental use of a connection is to take a *covariant derivative* ∇_μ (a generalization of the partial derivative); the covariant derivative of a vector field V^ν is given by

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^{\nu}_{\mu\sigma} V^\sigma, \tag{3.2}$$

and covariant derivatives of other sorts of tensors are given by similar expressions. The connection also appears in the definition of *geodesics* (a generalization of the notion of a straight line). A parameterized curve $x^\mu(\lambda)$ is a geodesic if it obeys

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^{\mu}_{\rho\sigma} \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda} = 0, \tag{3.3}$$

known as the geodesic equation.

Finally, the technical expression of curvature is contained in the Riemann tensor, a (1, 3) tensor obtained from the connection by

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}. \tag{3.4}$$

Everything we want to know about the curvature of a manifold is given to us by the Riemann tensor; it will vanish if and only if the metric is perfectly flat. Einstein's equation of general relativity relates certain components of this tensor to the energy-momentum tensor.

These four equations are all of primary importance in the study of curved manifolds. We will now see how they arise from a careful consideration of how familiar notions of geometry in flat space adapt to this more general context.

3.2 ■ COVARIANT DERIVATIVES

In our discussion of manifolds, it became clear that there were various notions we could talk about as soon as the manifold was defined: we could define functions, take their derivatives, consider parameterized paths, set up tensors, and so on. Other concepts, such as the volume of a region or the length of a path, required some additional piece of structure, namely the introduction of a metric. It would be natural to think of the notion of curvature as something that depends exclusively on the metric. In a more careful treatment, however, we find that curvature depends on a connection, and connections may or may not depend on the metric. Nevertheless, we will also show how the existence of a metric implies a certain unique connection, whose curvature may be thought of as that of the metric. This is the connection used in general relativity, so in this particular context it is legitimate to think of curvature as characterizing the metric, without introducing any additional structures.

The connection becomes necessary when we attempt to address the problem of the partial derivative not being a good tensor operator. What we would like is a covariant derivative, that is, an operator that reduces to the partial derivative in flat space with inertial coordinates, but transforms as a tensor on an arbitrary manifold. It is conventional to spend a certain amount of time motivating the introduction of a covariant derivative, but in fact the need is obvious; equations such as $\partial_\mu T^{\mu\nu} = 0$ must be generalized to curved space somehow. So let's agree that a covariant derivative would be a good thing to have, and go about setting it up.

In flat space in inertial coordinates, the partial derivative operator ∂_μ is a map from (k, l) tensor fields to $(k, l + 1)$ tensor fields, which acts linearly on its arguments and obeys the Leibniz rule on tensor products. All of this continues to be true in the more general situation we would now like to consider, but the map provided by the partial derivative depends on the coordinate system used. We would therefore like to define a **covariant derivative** operator ∇ to perform the functions of the partial derivative, but in a way independent of coordinates. Rather than simply postulating the answer (which would be perfectly acceptable), let's

motivate it by thinking carefully about what properties a covariant generalization of the partial derivative *should* have—mathematical structures are, after all, invented by human beings, not found lying on sidewalks. We begin by requiring that ∇ be a map from (k, l) tensor fields, to $(k, l+1)$ tensor fields which has these two properties:

1. linearity: $\nabla(T + S) = \nabla T + \nabla S$;
2. Leibniz (product) rule: $\nabla(T \otimes S) = (\nabla T) \otimes S + T \otimes (\nabla S)$.

If ∇ is going to obey the Leibniz rule, it can always be written as the partial derivative plus some linear transformation. That is, to take the covariant derivative we first take the partial derivative, and then apply a correction to make the result covariant. [We aren't going to prove this reasonable-sounding statement; see Wald (1984) if you are interested.] Let's consider what this means for the covariant derivative of a vector V^ν. It means that, for each direction μ, the covariant derivative ∇_μ will be given by the partial derivative ∂_μ plus a correction specified by a set of n matrices $(\Gamma_\mu)^\rho{}_\sigma$ (one $n \times n$ matrix, where n is the dimensionality of the manifold, for each μ). In fact the parentheses are usually dropped and we write these matrices, known as the **connection coefficients**, with haphazard index placement as $\Gamma^\rho_{\mu\sigma}$. We therefore have

$$\boxed{\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda.} \tag{3.5}$$

Notice that in the second term the index originally on V has moved to the Γ, and a new index is summed over. If this is the expression for the covariant derivative of a vector in terms of the partial derivative, we should be able to determine the transformation properties of $\Gamma^\nu_{\mu\lambda}$ by demanding that the left-hand side be a (1, 1) tensor. That is, we want the transformation law to be

$$\nabla_{\mu'} V^{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \nabla_\mu V^\nu. \tag{3.6}$$

Let's look at the left side first; we can expand it using (3.5) and then transform the parts that we understand (which is everything except $\Gamma^{\nu'}_{\mu'\lambda'}$):

$$\begin{aligned} \nabla_{\mu'} V^{\nu'} &= \partial_{\mu'} V^{\nu'} + \Gamma^{\nu'}_{\mu'\lambda'} V^{\lambda'} \\ &= \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \partial_\mu V^\nu + \frac{\partial x^\mu}{\partial x^{\mu'}} V^\nu \frac{\partial}{\partial x^\mu} \frac{\partial x^{\nu'}}{\partial x^\nu} + \Gamma^{\nu'}_{\mu'\lambda'} \frac{\partial x^{\lambda'}}{\partial x^\lambda} V^\lambda. \end{aligned} \tag{3.7}$$

On the right-hand side we can also expand $\nabla_\mu V^\nu$:

$$\frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \nabla_\mu V^\nu = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \partial_\mu V^\nu + \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \Gamma^\nu_{\mu\lambda} V^\lambda. \tag{3.8}$$

These last two expressions are to be equated; the first terms in each are identical and therefore cancel, so we have

$$\Gamma^{\nu'}_{\mu'\lambda'}\frac{\partial x^{\lambda'}}{\partial x^{\lambda}}V^{\lambda}+\frac{\partial x^{\mu}}{\partial x^{\mu'}}V^{\lambda}\frac{\partial}{\partial x^{\mu}}\frac{\partial x^{\nu'}}{\partial x^{\lambda}}=\frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^{\nu}}\Gamma^{\nu}_{\mu\lambda}V^{\lambda}, \tag{3.9}$$

where we have changed a dummy index from ν to λ. This equation must be true for any vector V^{λ}, so we can eliminate that on both sides. Then the connection coefficients in the primed coordinates may be isolated by multiplying by $\partial x^{\lambda}/\partial x^{\sigma'}$ and relabeling $\sigma' \to \lambda'$. The result is

$$\Gamma^{\nu'}_{\mu'\lambda'}=\frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\lambda}}{\partial x^{\lambda'}}\frac{\partial x^{\nu'}}{\partial x^{\nu}}\Gamma^{\nu}_{\mu\lambda}+\frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\lambda}}{\partial x^{\lambda'}}\frac{\partial^2 x^{\nu'}}{\partial x^{\mu}\partial x^{\lambda}}. \tag{3.10}$$

This is not, of course, the tensor transformation law; the second term on the right spoils it. That's okay, because *the connection coefficients are not the components of a tensor*. They are purposefully constructed to be nontensorial, but in such a way that the combination (3.5) transforms as a tensor—the extra terms in the transformation of the partials and the Γ's exactly cancel. This is why we are not so careful about index placement on the connection coefficients; they are not a tensor, and therefore you should try not to raise and lower their indices.

What about the covariant derivatives of other sorts of tensors? By similar reasoning to that used for vectors, the covariant derivative of a one-form can also be expressed as a partial derivative plus some linear transformation. But there is no reason as yet that the matrices representing this transformation should be related to the coefficients $\Gamma^{\nu}_{\mu\lambda}$. In general we could write something like

$$\nabla_{\mu}\omega_{\nu}=\partial_{\mu}\omega_{\nu}+\widetilde{\Gamma}^{\lambda}_{\mu\nu}\omega_{\lambda}, \tag{3.11}$$

where $\widetilde{\Gamma}^{\lambda}_{\mu\nu}$ is a new set of matrices for each μ. Pay attention to where all of the various indices go. It is straightforward to derive that the transformation properties of $\widetilde{\Gamma}$ must be similar to those of Γ, since otherwise $\nabla_{\mu}\omega_{\nu}$ wouldn't transform as a tensor, but otherwise no relationship has been established. To do so, we need to introduce two new properties that we would like our covariant derivative to have, in addition to the two above:

3. commutes with contractions: $\nabla_{\mu}(T^{\lambda}{}_{\lambda\rho})=(\nabla T)_{\mu}{}^{\lambda}{}_{\lambda\rho}$.
4. reduces to the partial derivative on scalars: $\nabla_{\mu}\phi=\partial_{\mu}\phi$.

There is no way to "derive" these properties; we are simply demanding that they be true as part of the definition of a covariant derivative. Note that property 3 is equivalent to saying that the Kronecker delta (the identity map) is covariantly constant, $\nabla_{\mu}\delta^{\lambda}_{\sigma}=0$; this is certainly a reasonable thing to ask.

Let's see what these new properties imply. Given some one-form field ω_{μ} and vector field V^{μ}, we can take the covariant derivative of the scalar defined by $\omega_{\lambda}V^{\lambda}$ to get

$$\begin{aligned}\nabla_\mu(\omega_\lambda V^\lambda) &= (\nabla_\mu \omega_\lambda)V^\lambda + \omega_\lambda(\nabla_\mu V^\lambda) \\ &= (\partial_\mu \omega_\lambda)V^\lambda + \widetilde{\Gamma}^\sigma_{\mu\lambda}\omega_\sigma V^\lambda + \omega_\lambda(\partial_\mu V^\lambda) + \omega_\lambda \Gamma^\lambda_{\mu\rho} V^\rho. \end{aligned} \tag{3.12}$$

But since $\omega_\lambda V^\lambda$ is a scalar, this must also be given by the partial derivative:

$$\begin{aligned}\nabla_\mu(\omega_\lambda V^\lambda) &= \partial_\mu(\omega_\lambda V^\lambda) \\ &= (\partial_\mu \omega_\lambda)V^\lambda + \omega_\lambda(\partial_\mu V^\lambda). \end{aligned} \tag{3.13}$$

This can only be true if the terms in (3.12) with connection coefficients cancel each other; that is, rearranging dummy indices, we must have

$$0 = \widetilde{\Gamma}^\sigma_{\mu\lambda}\omega_\sigma V^\lambda + \Gamma^\sigma_{\mu\lambda}\omega_\sigma V^\lambda. \tag{3.14}$$

But both ω_σ and V^λ are completely arbitrary, so

$$\widetilde{\Gamma}^\sigma_{\mu\lambda} = -\Gamma^\sigma_{\mu\lambda}. \tag{3.15}$$

The two extra conditions we have imposed therefore allow us to express the covariant derivative of a one-form using the same connection coefficients as were used for the vector, but now with a minus sign (and indices matched up somewhat differently):

$$\boxed{\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\lambda_{\mu\nu}\omega_\lambda.} \tag{3.16}$$

It should come as no surprise that the connection coefficients encode all the information necessary to take the covariant derivative of a tensor of arbitrary rank. The formula is quite straightforward; for each upper index you introduce a term with a single $+\Gamma$, and for each lower index a term with a single $-\Gamma$:

$$\begin{aligned}\nabla_\sigma T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} &= \partial_\sigma T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} \\ &\quad + \Gamma^{\mu_1}_{\sigma\lambda} T^{\lambda\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} + \Gamma^{\mu_2}_{\sigma\lambda} T^{\mu_1\lambda\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} + \cdots \\ &\quad - \Gamma^\lambda_{\sigma\nu_1} T^{\mu_1\mu_2\cdots\mu_k}{}_{\lambda\nu_2\cdots\nu_l} - \Gamma^\lambda_{\sigma\nu_2} T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\lambda\cdots\nu_l} - \cdots. \end{aligned} \tag{3.17}$$

This is the general expression for the covariant derivative. You can check it yourself; it comes from the set of axioms we have established, and the usual requirements that tensors of various sorts be coordinate-independent entities. Sometimes an alternative notation is used; just as commas are used for partial derivatives, semicolons are used for covariant ones:

$$\nabla_\sigma T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} \equiv T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l;\sigma}. \tag{3.18}$$

Once again, in this book we will stick to "∇_σ."

To define a covariant derivative, then, we need to put a connection on our manifold, which is specified in some coordinate system by a set of coefficients $\Gamma^{\lambda}_{\mu\nu}$ ($n^3 = 64$ independent components in $n = 4$ dimensions) that transform according to (3.10). (The name *connection* comes from the fact that it is used to transport vectors from one tangent space to another, as we will soon see; it is sometimes used to refer to the operator ∇, and sometimes to the coefficients $\Gamma^{\lambda}_{\mu\nu}$.) Evidently, we could define a large number of connections on any manifold, and each of them implies a distinct notion of covariant differentiation. In general relativity this freedom is not a big concern, because it turns out that every metric defines a unique connection, which is the one used in GR. Let's see how that works.

The first thing to notice is that the difference of two connections is a tensor. Imagine we have defined two different kinds of covariant derivative, ∇_μ and $\widehat{\nabla}_\mu$, with associated connection coefficients $\Gamma^{\lambda}_{\mu\nu}$ and $\widehat{\Gamma}^{\lambda}_{\mu\nu}$. Then the difference

$$S^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} - \widehat{\Gamma}^{\lambda}_{\mu\nu} \tag{3.19}$$

is a (1, 2) tensor. (Notice that we had to choose a convention for index placement.) We could show this by brute force, plugging in the transformation laws for the connection coefficients, but let's be a little more slick. Given an arbitrary vector field V^λ, we know that both $\nabla_\mu V^\lambda$ and $\widehat{\nabla}_\mu V^\lambda$ are tensors, so their difference must also be. This difference is simply

$$\begin{aligned} \nabla_\mu V^\lambda - \widehat{\nabla}_\mu V^\lambda &= \partial_\mu V^\lambda + \Gamma^{\lambda}_{\mu\nu} V^\nu - \partial_\mu V^\lambda - \widehat{\Gamma}^{\lambda}_{\mu\nu} V^\nu \\ &= S^{\lambda}{}_{\mu\nu} V^\nu . \end{aligned} \tag{3.20}$$

Since V^λ was arbitrary, and the left-hand side is a tensor, $S^{\lambda}{}_{\mu\nu}$ must be a tensor. As a trivial consequence, we learn that any set of connection coefficients can be expressed as some fiducial connection plus a tensorial correction,

$$\Gamma^{\lambda}_{\mu\nu} = \widehat{\Gamma}^{\lambda}_{\mu\nu} + S^{\lambda}{}_{\mu\nu}. \tag{3.21}$$

Next notice that, given a connection specified by $\Gamma^{\lambda}_{\mu\nu}$, we can immediately form another connection simply by permuting the lower indices. That is, the set of coefficients $\Gamma^{\lambda}_{\nu\mu}$ will also transform according to (3.10) (since the partial derivatives appearing in the last term can be commuted), so they determine a distinct connection. There is thus a tensor we can associate with any given connection, known as the **torsion tensor**, defined by

$$T^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu} = 2\Gamma^{\lambda}_{[\mu\nu]}. \tag{3.22}$$

It is clear that the torsion is antisymmetric in its lower indices, and a connection that is symmetric in its lower indices is known as "torsion-free."

We can now define a unique connection on a manifold with a metric $g_{\mu\nu}$ by introducing two additional properties:

- torsion-free: $\Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{(\mu\nu)}$.
- metric compatibility: $\nabla_\rho g_{\mu\nu} = 0$.

A connection is **metric compatible** if the covariant derivative of the metric with respect to that connection is everywhere zero. This implies a couple of nice properties. First, it's easy to show that both the Levi–Civita tensor and the inverse metric also have zero covariant derivative,

$$\begin{aligned} \nabla_\lambda \epsilon_{\mu\nu\rho\sigma} &= 0 \\ \nabla_\rho g^{\mu\nu} &= 0. \end{aligned} \tag{3.23}$$

Second, a metric-compatible covariant derivative commutes with raising and lowering of indices. Thus, for some vector field V^λ,

$$g_{\mu\lambda}\nabla_\rho V^\lambda = \nabla_\rho(g_{\mu\lambda}V^\lambda) = \nabla_\rho V_\mu. \tag{3.24}$$

With non-metric-compatible connections we would have to be very careful about index placement when taking a covariant derivative.

Our claim is therefore that there is exactly one torsion-free connection on a given manifold that is compatible with some given metric on that manifold. We do not want to make these two requirements part of the definition of a covariant derivative; they simply single out one of the many possible ones.

We can demonstrate both existence and uniqueness by deriving a manifestly unique expression for the connection coefficients in terms of the metric. To accomplish this, we expand out the equation of metric compatibility for three different permutations of the indices:

$$\begin{aligned} \nabla_\rho g_{\mu\nu} &= \partial_\rho g_{\mu\nu} - \Gamma^{\lambda}_{\rho\mu} g_{\lambda\nu} - \Gamma^{\lambda}_{\rho\nu} g_{\mu\lambda} = 0 \\ \nabla_\mu g_{\nu\rho} &= \partial_\mu g_{\nu\rho} - \Gamma^{\lambda}_{\mu\nu} g_{\lambda\rho} - \Gamma^{\lambda}_{\mu\rho} g_{\nu\lambda} = 0 \\ \nabla_\nu g_{\rho\mu} &= \partial_\nu g_{\rho\mu} - \Gamma^{\lambda}_{\nu\rho} g_{\lambda\mu} - \Gamma^{\lambda}_{\nu\mu} g_{\rho\lambda} = 0. \end{aligned} \tag{3.25}$$

We subtract the second and third of these from the first, and use the symmetry of the connection to obtain

$$\partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho} - \partial_\nu g_{\rho\mu} + 2\Gamma^{\lambda}_{\mu\nu} g_{\lambda\rho} = 0. \tag{3.26}$$

It is straightforward to solve this for the connection by multiplying by $g^{\sigma\rho}$. The result is

$$\boxed{\Gamma^{\sigma}_{\mu\nu} = \tfrac{1}{2} g^{\sigma\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}).} \tag{3.27}$$

This formula is one of the most important in this subject; commit it to memory. Of course, we have only proved that if a metric-compatible and torsion-free connection exists, it must be of the form (3.27); you can check for yourself that the right-hand side of (3.27) transforms like a connection.

This connection we have derived from the metric is the one on which conventional general relativity is based. It is known by different names: sometimes the **Christoffel** connection, sometimes the **Levi–Civita** connection, sometimes the **Riemannian** connection. The associated connection coefficients are sometimes called **Christoffel symbols** and written as $\left\{ {\sigma \atop \mu\nu} \right\}$; we will sometimes call them Christoffel symbols, but we won't use the funny notation. The study of manifolds with metrics and their associated connections is called Riemannian geometry, or sometimes pseudo-Riemannian when the metric has a Lorentzian signature.

Before putting our covariant derivatives to work, we should mention some miscellaneous properties. First, note that in ordinary flat space there is an implicit connection we use all the time—the Christoffel connection constructed from the flat metric. The coefficients of the Christoffel connection in flat space vanish in Cartesian coordinates, but not in curvilinear coordinate systems. Consider for example the plane in polar coordinates, with metric

$$ds^2 = \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2. \tag{3.28}$$

The nonzero components of the inverse metric are readily found to be $g^{rr} = 1$ and $g^{\theta\theta} = r^{-2}$. Notice that we use r and θ as indices in an obvious notation. We can compute a typical connection coefficient:

$$\begin{aligned}
\Gamma^r_{rr} &= \tfrac{1}{2} g^{r\rho} (\partial_r g_{r\rho} + \partial_r g_{\rho r} - \partial_\rho g_{rr}) \\
&= \tfrac{1}{2} g^{rr} (\partial_r g_{rr} + \partial_r g_{rr} - \partial_r g_{rr}) \\
&\quad + \tfrac{1}{2} g^{r\theta} (\partial_r g_{r\theta} + \partial_r g_{\theta r} - \partial_\theta g_{rr}) \\
&= \tfrac{1}{2}(1)(0 + 0 - 0) + \tfrac{1}{2}(0)(0 + 0 - 0) \\
&= 0.
\end{aligned} \tag{3.29}$$

Sadly, it vanishes. But not all of them do:

$$\begin{aligned}
\Gamma^r_{\theta\theta} &= \tfrac{1}{2} g^{r\rho} (\partial_\theta g_{\theta\rho} + \partial_\theta g_{\rho\theta} - \partial_\rho g_{\theta\theta}) \\
&= \tfrac{1}{2} g^{rr} (\partial_\theta g_{\theta r} + \partial_\theta g_{r\theta} - \partial_r g_{\theta\theta}) \\
&= \tfrac{1}{2}(1)(0 + 0 - 2r) \\
&= -r.
\end{aligned} \tag{3.30}$$

Continuing to turn the crank, we eventually find

$$\begin{aligned}
&\Gamma^r_{\theta r} = \Gamma^r_{r\theta} = 0 \\
&\Gamma^\theta_{rr} = 0 \\
&\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r} \\
&\Gamma^\theta_{\theta\theta} = 0.
\end{aligned} \tag{3.31}$$

From these and similar expressions, we can derive formulas for the divergence, gradient, and curl in curvilinear coordinate systems.

Contrariwise, even in a curved space it is still possible to make the Christoffel symbols vanish at any one point. This is because, as we argued in the last chapter, we can always make the first derivative of the metric vanish at a point; so by (3.27) the connection coefficients derived from this metric will also vanish. Of course this can only be established at a point, not in some neighborhood of the point. We will discuss this more fully in Section 3.4.

Another useful property is that the formula for the divergence of a vector (with respect to the Christoffel connection) has a simplified form. The covariant divergence of V^μ is given by

$$\nabla_\mu V^\mu = \partial_\mu V^\mu + \Gamma^\mu_{\mu\lambda} V^\lambda. \tag{3.32}$$

It is straightforward to show that the Christoffel connection satisfies

$$\Gamma^\mu_{\mu\lambda} = \frac{1}{\sqrt{|g|}} \partial_\lambda \sqrt{|g|}, \tag{3.33}$$

and we therefore obtain

$$\nabla_\mu V^\mu = \frac{1}{\sqrt{|g|}} \partial_\mu (\sqrt{|g|} V^\mu). \tag{3.34}$$

There are also formulas for the divergences of higher-rank tensors, but they are generally not such a great simplification.

We use the Christoffel covariant derivative in the curved-space version of Stokes's theorem (see Appendix E). If V^μ is a vector field over a region Σ with boundary $\partial\Sigma$, Stokes's theorem is

$$\boxed{\int_\Sigma \nabla_\mu V^\mu \sqrt{|g|}\, d^n x = \int_{\partial\Sigma} n_\mu V^\mu \sqrt{|\gamma|}\, d^{n-1} x,} \tag{3.35}$$

where n_μ is normal to $\partial\Sigma$, and γ_{ij} is the induced metric on $\partial\Sigma$. If the connection weren't metric-compatible or torsion-free, there would be additional terms in this equation.

The last thing we need to mention is that converting partial derivatives into covariant derivatives is not always necessary in order to construct well-defined tensors; in particular, the exterior derivative and the vector-field commutator are both well-defined in terms of partials, essentially because both involve an antisymmetrization that cancels the nontensorial piece of the partial derivative transformation law. The same feature implies that they could, on the other hand, be equally well-defined in terms of (torsion-free) covariant derivatives; the antisymmetrization causes the connection coefficient terms to vanish. Thus, if ∇ is the Christoffel connection, ω_μ is a one-form, and X^μ and Y^μ are vector fields, we

can write

$$(\mathrm{d}\omega)_{\mu\nu} = 2\partial_{[\mu}\omega_{\nu]} = 2\nabla_{[\mu}\omega_{\nu]} \tag{3.36}$$

and

$$[X, Y]^\mu = X^\lambda \partial_\lambda Y^\mu - Y^\lambda \partial_\lambda X^\mu = X^\lambda \nabla_\lambda Y^\mu - Y^\lambda \nabla_\lambda X^\mu. \tag{3.37}$$

If the connection is not torsion-free, the last equalities in these expressions are no longer true; the more fundamental definitions of the exterior derivative and the commutator are those in terms of the partial derivative.

Before moving on, let's review the process by which we have been adding structures to our mathematical constructs. We started with the basic notion of a set, which you were presumed to be familiar with (informally, if not rigorously). We introduced the concept of open subsets of our set; this is equivalent to introducing a topology, and promoted the set to a topological space. Then by demanding that each open set look like a region of $\mathbf{R}^n$ (with n the same for each set) and that the coordinate charts be smoothly sewn together, the topological space became a manifold. A manifold is simultaneously a very flexible and powerful structure, and comes equipped naturally with a tangent bundle, tensor bundles of various ranks, the ability to take exterior derivatives, and so forth. We then proceeded to put a metric on the manifold, resulting in a manifold with metric (sometimes Riemannian manifold). Independently of the metric we found we could introduce a connection, allowing us to take covariant derivatives. Once we have a metric, however, there is automatically a unique torsion-free metric-compatible connection. Likewise we could introduce an independent volume form, although one is automatically determined by the metric. In principle there is nothing to stop us from introducing more than one connection, or volume form, or metric, on any given manifold. In general relativity we do have a physical metric, which determines volumes and the covariant derivative, and the independence of these notions is not a crucial feature.

3.3 ■ PARALLEL TRANSPORT AND GEODESICS

Now that we know how to take covariant derivatives, let's step back and put this in the context of differentiation more generally. We think of a derivative as a way of quantifying how fast something is changing. In the case of tensors, the crucial issue is "changing with respect to what?" An ordinary function defines a number at each point in spacetime, and it is straightforward to compare two different numbers, so we shouldn't be surprised that the partial derivative of functions remained valid on arbitrary manifolds. But a tensor is a map from vectors and dual vectors to the real numbers, and it's not clear how to compare such maps at different points in spacetime. Since we have successfully constructed a covariant derivative, can we think of it as somehow measuring the rate of change of tensors? The answer turns out to be yes: the covariant derivative quantifies the instantaneous rate of

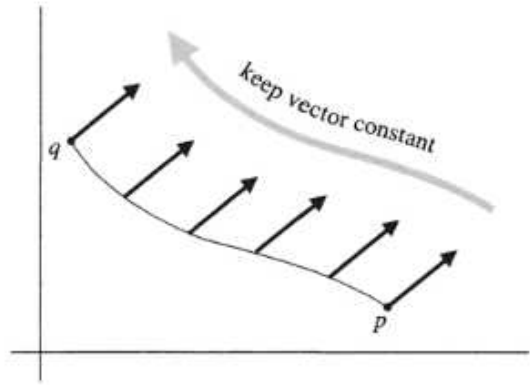


FIGURE 3.1 In flat space, we can parallel transport a vector by simply keeping its Cartesian components constant.

change of a tensor field in comparison to what the tensor would be if it were "parallel transported." In other words, a connection defines a specific way of keeping a tensor constant (along some path), on the basis of which we can compare nearby tensors.

It turns out that the concept of parallel transport is interesting in its own right, and worth spending some time thinking about. Recall that in flat space it is unnecessary to be very careful about the fact that vectors are elements of tangent spaces defined at individual points; it is actually very natural to compare vectors at different points (where by "compare" we mean add, subtract, take the dot product, and so on). The reason why it is natural is because it makes sense, in flat space, to move a vector from one point to another while keeping it constant, as illustrated in Figure 3.1. Then, once we get the vector from one point to another, we can do the usual operations allowed in a vector space.

This concept of moving a vector along a path, keeping constant all the while, is known as *parallel transport*. Parallel transport requires a connection to be well-defined; the intuitive manipulation of vectors in flat space makes implicit use of the Christoffel connection on this space. The crucial difference between flat and curved spaces is that, in a curved space, *the result of parallel transporting a vector from one point to another will depend on the path taken between the points.* Without yet assembling the complete mechanism of parallel transport, we can use our intuition about the two-sphere to see that this is the case. Start with a vector on the equator, pointing along a line of constant longitude. Parallel transport it up to the north pole along a line of longitude in the obvious way. Then take the original vector, parallel transport it along the equator by an angle θ, and then move it up to the north pole as before. As shown in Figure 3.2, it should be clear that the vector, parallel transported along two paths, arrived at the same destination with two different values (rotated by θ).

It therefore appears as if there is no natural way to uniquely move a vector from one tangent space to another; we can always parallel-transport it, but the result depends on the path, and there is no natural choice of which path to take.

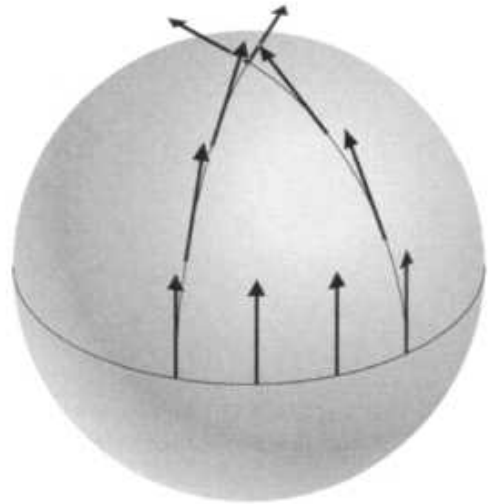


FIGURE 3.2 Parallel transport on a two-sphere. On a curved manifold, the result of parallel transport can depend on the path taken.

Unlike some of the problems we have encountered, *there is no solution to this one*—we simply must learn to live with the fact that two vectors can only be compared in a natural way if they are elements of the same tangent space. For example, two particles passing by each other have a well-defined relative velocity, which cannot be greater than the speed of light. But two particles at different points on a curved manifold do not have any well-defined notion of relative velocity—the concept simply makes no sense. Of course, in certain special situations it is still useful to talk as if it did make sense, but occasional usefulness is not a substitute for rigorous definition. In cosmology, for example, the light from distant galaxies is redshifted with respect to the frequencies we would observe from a nearby stationary source. Since this phenomenon bears such a close resemblance to the conventional Doppler effect due to relative motion, we are very tempted to say that the galaxies are "receding away from us" at a speed defined by their redshift. At a rigorous level this is nonsense, what Wittgenstein would call a "grammatical mistake"—the galaxies are not receding, since the notion of their velocity with respect to us is not well-defined. What is actually happening is that the metric of spacetime between us and the galaxies has changed (the universe has expanded) along the path of the photon from here to there, leading to an increase in the wavelength of the light. As an example of how you can go wrong, naive application of the Doppler formula to the redshift of galaxies implies that some of them are receding faster than light, in apparent contradiction with relativity. The resolution of this apparent paradox is simply that the very notion of their recession should not be taken literally.

Enough about what we cannot do; let's see what we can. Parallel transport is supposed to be the curved-space generalization of the concept of "keeping the vector constant" as we move it along a path; similarly for a tensor of arbitrary rank.

Given a curve $x^\mu(\lambda)$, the requirement of constancy of a tensor $T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l}$ along this curve in flat space is simply that the components be constant:

$$\frac{d}{d\lambda}T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} = \frac{dx^\mu}{d\lambda}\frac{\partial}{\partial x^\mu}T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} = 0.$$

To make this properly tensorial we simply replace this partial derivative by a covariant one, and define the **directional covariant derivative** to be

$$\frac{D}{d\lambda} = \frac{dx^\mu}{d\lambda}\nabla_\mu. \tag{3.38}$$

This is a map, defined only along the path, from (k, l) tensors to (k, l) tensors. We then define **parallel transport** of the tensor T along the path $x^\mu(\lambda)$ to be the requirement that the covariant derivative of T along the path vanishes:

$$\left(\frac{D}{d\lambda}T\right)^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} \equiv \frac{dx^\sigma}{d\lambda}\nabla_\sigma T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} = 0. \tag{3.39}$$

This is a well-defined tensor equation (since both the tangent vector $dx^\mu/d\lambda$ and the covariant derivative ∇T are tensors), known as the **equation of parallel transport**. For a vector it takes the form

$$\frac{d}{d\lambda}V^\mu + \Gamma^\mu_{\sigma\rho}\frac{dx^\sigma}{d\lambda}V^\rho = 0. \tag{3.40}$$

We can look at the parallel transport equation as a first-order differential equation defining an initial-value problem: given a tensor at some point along the path, there will be a unique continuation of the tensor to other points along the path such that the continuation solves (3.39). We say that such a tensor is parallel-transported.

The notion of parallel transport is obviously dependent on the connection, and different connections lead to different answers. If the connection is metric-compatible, the metric is always parallel transported with respect to it:

$$\frac{D}{d\lambda}g_{\mu\nu} = \frac{dx^\sigma}{d\lambda}\nabla_\sigma g_{\mu\nu} = 0. \tag{3.41}$$

It follows that the inner product of two parallel-transported vectors is preserved. That is, if V^μ and W^ν are parallel-transported along a curve $x^\sigma(\lambda)$, we have

$$\begin{aligned}\frac{D}{d\lambda}(g_{\mu\nu}V^\mu W^\nu) &= \left(\frac{D}{d\lambda}g_{\mu\nu}\right)V^\mu W^\nu + g_{\mu\nu}\left(\frac{D}{d\lambda}V^\mu\right)W^\nu + g_{\mu\nu}V^\mu\left(\frac{D}{d\lambda}W^\nu\right)\\ &= 0.\end{aligned} \tag{3.42}$$

This means that parallel transport with respect to a metric-compatible connection preserves the norm of vectors, the sense of orthogonality, and so on.

With parallel transport defined, the next logical step is to discuss geodesics. A geodesic is the curved-space generalization of the notion of a straight line in Euclidean space. We all know what a straight line is: it's the path of shortest distance

between two points. But there is an equally good definition—a straight line is a path that parallel-transports its own tangent vector. It will turn out that these two concepts coincide if and only if the connection is the Christoffel connection.

We'll start with the second definition (a geodesic is a curve along which the tangent vector is parallel-transported), since it is computationally much more straightforward. The tangent vector to a path $x^\mu(\lambda)$ is $dx^\mu/d\lambda$. The condition that it be parallel transported is thus

$$\frac{D}{d\lambda}\frac{dx^\mu}{d\lambda} = 0, \tag{3.43}$$

or alternatively

$$\boxed{\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0.} \tag{3.44}$$

This is the **geodesic equation**, another one you should memorize. We can easily see that it reproduces the usual notion of straight lines if the connection coefficients are the Christoffel symbols in Euclidean space; in that case we can choose Cartesian coordinates in which $\Gamma^\mu_{\rho\sigma} = 0$, and the geodesic equation is just $d^2x^\mu/d\lambda^2 = 0$, which is the equation for a straight line.

That was embarrassingly simple; let's turn to the more nontrivial case of the shortest-distance definition. As we know, various subtleties are involved in the definition of distance in a Lorentzian spacetime; for null paths the distance is zero, for timelike paths it's more convenient to use the proper time. So in the name of simplicity let's do the calculation just for a timelike path—the resulting equation will turn out to be good for any path, so we are not losing any generality. We therefore consider the proper time functional,

$$\tau = \int \left(-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right)^{1/2} d\lambda, \tag{3.45}$$

where the integral is over the path. To search for shortest-distance paths, we could do the usual calculus-of-variations treatment to seek critical points of this functional. They will turn out to be curves of *maximum* proper time, consistent with our discussion of the twin paradox in Chapter 1. However, we can simplify the algebra by means of a trick. The integral (3.45) is of the form $\tau = \int \sqrt{-f}\, d\lambda$, where $f = g_{\mu\nu}(dx^\mu/d\lambda)(dx^\nu/d\lambda)$. The variation looks like

$$\begin{aligned}\delta\tau &= \int \delta\sqrt{-f}\, d\lambda \\ &= -\int \frac{1}{2}(-f)^{-1/2}\delta f\, d\lambda.\end{aligned} \tag{3.46}$$

It makes things easier if we now specify that our parameter is the proper time τ itself, rather than the arbitrary parameter λ, so that the tangent vector is the

four-velocity U^μ. This fixes the value of f,

$$f = g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = g_{\mu\nu}U^\mu U^\nu = -1. \tag{3.47}$$

From (3.46) we then have

$$\delta\tau = -\frac{1}{2}\int \delta f\, d\tau. \tag{3.48}$$

Stationary points of (3.45)—paths for which $\delta\tau = 0$—are therefore equivalent to stationary points (with fixed parameterization) of the simpler integral

$$I = \frac{1}{2}\int f\, d\tau = \frac{1}{2}\int g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\, d\tau. \tag{3.49}$$

(The $\frac{1}{2}$ is by no means necessary, but will make things nicer later on.) Taking variations of this expression is thus a shortcut to finding shortest-distance paths, one that we will wisely follow.

Stationary points of I will of course obey the Euler–Lagrange equations (1.128), but evaluating them involves repeated application of the chain rule, and it is just as simple to directly consider the change in the integral under infinitesimal variations of the path,

$$\begin{aligned} x^\mu &\to x^\mu + \delta x^\mu \\ g_{\mu\nu} &\to g_{\mu\nu} + (\partial_\sigma g_{\mu\nu})\delta x^\sigma. \end{aligned} \tag{3.50}$$

The second line comes from Taylor expansion in curved spacetime, which as you can see, uses the partial derivative, not the covariant derivative. This is because we are simply thinking of the components $g_{\mu\nu}$ as functions on spacetime in some specific coordinate system. Plugging this into (3.49) and keeping only terms first-order in δx^μ, we get

$$\delta I = \frac{1}{2}\int \left[\partial_\sigma g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\delta x^\sigma + g_{\mu\nu}\frac{d(\delta x^\mu)}{d\tau}\frac{dx^\nu}{d\tau} + g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{d(\delta x^\nu)}{d\tau}\right] d\tau. \tag{3.51}$$

The last two terms can be integrated by parts; for example,

$$\begin{aligned} \frac{1}{2}\int \left[g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{d(\delta x^\nu)}{d\tau}\right] d\tau &= -\frac{1}{2}\int \left[g_{\mu\nu}\frac{d^2x^\mu}{d\tau^2} + \frac{dg_{\mu\nu}}{d\tau}\frac{dx^\mu}{d\tau}\right]\delta x^\nu\, d\tau \\ &= -\frac{1}{2}\int \left[g_{\mu\nu}\frac{d^2x^\mu}{d\tau^2} + \partial_\sigma g_{\mu\nu}\frac{dx^\sigma}{d\tau}\frac{dx^\mu}{d\tau}\right]\delta x^\nu\, d\tau, \end{aligned} \tag{3.52}$$

where we have neglected boundary terms, which vanish because we take our variation δx^μ to vanish at the endpoints of the path. In the second line we have used

the chain rule on the derivative of $g_{\mu\nu}$. The variation (3.51) then becomes, after rearranging some dummy indices,

$$\delta I = -\int \left[g_{\mu\sigma} \frac{d^2x^\mu}{d\tau^2} + \frac{1}{2} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu} \right) \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right] \delta x^\sigma \, d\tau. \tag{3.53}$$

Since we are searching for stationary points, we want δI to vanish for any variation δx^σ; this implies

$$g_{\mu\sigma} \frac{d^2x^\mu}{d\tau^2} + \frac{1}{2} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu} \right) \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0, \tag{3.54}$$

and multiplying by the inverse metric $g^{\rho\sigma}$ finally leads to

$$\frac{d^2x^\rho}{d\tau^2} + \frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu} \right) \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0. \tag{3.55}$$

We see that this is precisely the geodesic equation (3.40), but with the specific choice of Christoffel connection (3.27). Thus, on a manifold with metric, extremals of the length functional are curves that parallel transport their tangent vector with respect to the Christoffel connection associated with that metric. It doesn't matter if any other connection is defined on the same manifold. Of course, in GR the Christoffel connection is the only one used, so the two notions are the same.

The variational principle provides a convenient way to actually calculate the Christoffel symbols for a given metric. Rather than simply plugging into (3.27), it is often less work to explicitly vary the integral (3.49), with the metric of interest substituted in for $g_{\mu\nu}$. An example of this procedure is shown in Section 3.5.

3.4 ■ PROPERTIES OF GEODESICS

The primary usefulness of geodesics in general relativity is that they are the paths followed by unaccelerated test particles. A **test particle** is a body that does not itself influence the geometry through which it moves—never perfectly true, but often an excellent approximation. This concept allows us to explore, for example, the properties of the gravitational field around the Sun, without worrying about the field of the planet whose motion we are considering. The geodesic equation can be thought of as the generalization of Newton's law $\mathbf{f} = m\mathbf{a}$, for the case $\mathbf{f} = 0$, to curved spacetime. It is also possible to introduce forces by adding terms to the right-hand side; in fact, looking back to the expression (1.106) for the Lorentz force in special relativity, it is natural to guess that

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau} = \frac{q}{m} F^\mu{}_\nu \frac{dx^\nu}{d\tau}. \tag{3.56}$$

We will talk about this more later, but in fact your guess would be correct.

We should say some more careful words about the parameterization of a geodesic path. When we presented the geodesic equation as the requirement that the tangent vector be parallel-transported, (3.44), we parameterized our path with some parameter λ, whereas when we found the formula (3.55) for the extremal of the spacetime interval, we wound up with a very specific parameterization, the proper time. Of course from the form of (3.55) it is clear that a transformation,

$$\tau \to \lambda = a\tau + b, \tag{3.57}$$

for some constants a and b, leaves the equation invariant. Any parameter related to the proper time in this way is called an **affine parameter**, and is just as good as the proper time for parameterizing a geodesic. What was hidden in our derivation of (3.44) was that *the demand that the tangent vector be parallel-transported actually constrains the parameterization of the curve*, specifically to one related to the proper time by (3.57). In other words, if you start at some point and with some initial direction, and then construct a curve by beginning to walk in that direction and keeping your tangent vector parallel transported, you will not only define a path in the manifold but also (up to linear transformations) define the parameter along the path.

Of course, there is nothing to stop you from using any other parameterization you like, but then (3.44) will not be satisfied. More generally you will satisfy an equation of the form

$$\frac{d^2x^\mu}{d\alpha^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\alpha}\frac{dx^\sigma}{d\alpha} = f(\alpha)\frac{dx^\mu}{d\alpha}, \tag{3.58}$$

for some parameter $\alpha(\lambda)$, where $f(\alpha)$ is related to the affine parameter by

$$f(\alpha) = -\left(\frac{d^2\alpha}{d\lambda^2}\right)\left(\frac{d\alpha}{d\lambda}\right)^{-2}. \tag{3.59}$$

Conversely, if (3.58) is satisfied along a curve you can always find an affine parameter $\lambda(\alpha)$ for which the geodesic equation (3.44) will be satisfied.

For timelike paths, we can write the geodesic equation in terms of the four-velocity $U^\mu = dx^\mu/d\tau$ as

$$U^\lambda \nabla_\lambda U^\mu = 0. \tag{3.60}$$

Similarly, in terms of the four-momentum $p^\mu = mU^\mu$, the geodesic equation is simply

$$p^\lambda \nabla_\lambda p^\mu = 0. \tag{3.61}$$

This relation expresses the idea that freely-falling particles keep moving in the direction in which their momenta are pointing.

For null paths, the proper time vanishes and τ is not an appropriate affine parameter. Nevertheless, it is still perfectly well-defined to ask whether a parameter-

ized path $x^\mu(\lambda)$ satisfies the geodesic equation (3.44). If a null path is a geodesic for some parameter λ, it will also be a geodesic for any other affine parameter of the form $a\lambda + b$. However, there is no preferred choice among these parameters like the proper time is for timelike paths. Once we choose a parameter at some point along the path, of course, there is a unique continuation to the rest of the path if we want to solve the geodesic equation. It is often convenient to choose the normalization of the affine parameter λ along a null geodesic such that $dx^\mu/d\lambda$ is equal to the momentum four-vector:

$$p^\mu = \frac{dx^\mu}{d\lambda}. \tag{3.62}$$

This is in contrast to timelike paths, where $dx^\mu/d\tau$ is the momentum per unit mass. Then an observer with four-velocity U^μ measures the energy of the particle (or equivalently the frequency, since we are setting $\hbar = 1$) to be

$$E = -p_\mu U^\mu. \tag{3.63}$$

This expression always tells us the energy of a particle with momentum p^μ as measured by an observer with four-velocity U^μ, whether p^μ is null or timelike; you can check it by going to locally inertial coordinates. (A caveat: this expression for E does not include potential energy, only the intrinsic energy from motion and inertia. In a general spacetime there will not be a well-defined notion of gravitational potential energy, although in special cases it does exist.)

An important property of geodesics in a spacetime with Lorentzian metric is that the character (timelike/null/spacelike) of the geodesic, relative to a metric-compatible connection, never changes. This is simply because parallel transport preserves inner products, and the character is determined by the inner product of the tangent vector with itself. This is why we were consistent to consider purely timelike paths when we derived (3.55); for spacelike paths we would have derived the same equation, since the only difference is an overall minus sign in the final answer.

Let's now explain the earlier remark that timelike geodesics are maxima of the proper time. The reason we know this is true is that, given any timelike curve (geodesic or not), we can approximate it to arbitrary accuracy by a null curve. To do this all we have to do is to consider "jagged" null curves that follow the timelike one, as portrayed in Figure 3.3. As we increase the number of sharp corners, the null curve comes closer and closer to the timelike curve while still having zero path length. Timelike geodesics cannot therefore be curves of minimum proper time, since they are always infinitesimally close to curves of less proper time (zero, in fact); actually they maximize the proper time. This is how you can remember which twin in the twin paradox ages more—the one who stays home is basically on a geodesic, and therefore experiences more proper time. Of course even this is being a little cavalier; actually every time we say "maximize" or "minimize" we should add the modifier "locally." Often the case is that between two points on a manifold there is more than one geodesic. For instance, on S^2 we can

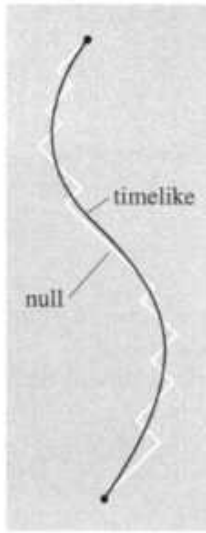


FIGURE 3.3 We can always approximate a timelike path by a sequence of null paths with a total path length of zero. Hence, timelike geodesics must be maxima of the proper time rather than minima.

draw a great circle through any two points, and imagine traveling between them either the short way or the long way around. One of these is obviously longer than the other, although both are stationary points of the length functional.

Geodesics provide a convenient way of mapping the tangent space T_p of a point p to a region of the manifold that contains p, called the **exponential map**. This map in turn defines a set of coordinates for this region that are automatically the locally inertial coordinates discussed in Section 2.5 [coordinates $x^{\hat{\mu}}$ around a point p such that $g_{\hat{\mu}\hat{\nu}}(p) = \eta_{\hat{\mu}\hat{\nu}}$ and $\partial_{\hat{\sigma}} g_{\hat{\mu}\hat{\nu}}(p) = 0$]. We begin by noticing that any vector $k \in T_p$ defines a unique geodesic passing through it, for which k is the tangent vector at p, and $\lambda(p) = 0$:

$$\frac{dx^\mu}{d\lambda}(\lambda = 0) = k^\mu. \tag{3.64}$$

Uniqueness follows from the fact that the geodesic equation is a second-order differential equation, and specifying initial data in the form $x^\mu(p)$ and $k^\mu = (dx^\mu/d\lambda)(p)$ completely determines a solution. On this geodesic there will be a unique point in M for which $\lambda = 1$. The exponential map at p, $\exp_p : T_p \to M$, is then defined as

$$\exp_p(k) = x^\nu(\lambda = 1), \tag{3.65}$$

where $x^\nu(\lambda)$ solves the geodesic equation subject to (3.64), as shown in Figure 3.4.

For some set of tangent vectors k^μ near the zero vector, this map will be well-defined, and in fact invertible. Depending on the geometry, however, different geodesics emanating from a single point may eventually cross, at which point $\exp_p : T_p \to M$ is no longer one-to-one. Furthermore, the range of the exponential map is not necessarily the whole manifold, and the domain is not necessarily the whole tangent space. The range can fail to be all of M simply because there can be two points that are not connected by any geodesic. An example is given by anti-de Sitter space, discussed in Chapter 8. The domain can fail to be all of T_p because a geodesic may run into a singularity, which we think of as "the edge of the manifold." Manifolds that have such singularities are known as **geodesically incomplete**. In a more careful discussion it would actually be the

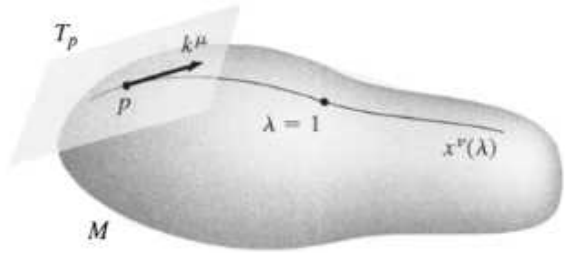


FIGURE 3.4 The exponential map takes a vector in T_p to a point in M that lies at unit affine parameter along the geodesic to which the vector is tangent.

other way around: the best way we have of defining a singularity is as a place where geodesics appear to "end," after we remove trivial cases in which a part of the manifold is artificially excluded by hand. See Wald (1984) or Hawking and Ellis (1973). This problem is not merely technical; the singularity theorems of Hawking and Penrose state that, for certain matter content, spacetimes in general relativity are almost guaranteed to be geodesically incomplete. As examples, two of the most useful spacetimes in GR—the Schwarzschild solution describing black holes and the Friedmann–Robertson–Walker solutions describing homogeneous, isotropic cosmologies—both feature important singularities; these will be discussed in later chapters.

We now use the exponential map to construct locally inertial coordinates. The easy part is to find basis vectors $\{\hat{e}_{(\hat{\mu})}\}$ for T_p such that the components of the metric are those of the canonical form:

$$g_{\hat{\mu}\hat{\nu}} = g(\hat{e}_{(\hat{\mu})}, \hat{e}_{(\hat{\nu})}) = \eta_{\hat{\mu}\hat{\nu}}. \tag{3.66}$$

Here $g(\ ,\ )$ denotes the metric, thought of as a multilinear map from $T_p \times T_p$ to $\mathbf{R}$. And the hats have different meanings: over e they remind us that it's a basis vector, and over the indices they remind us that we are in locally inertial coordinates (as we shall see). This is easy because it's just linear algebra, not yet referring to coordinates; starting with any set of components for $g_{\mu\nu}$, we can always diagonalize this matrix and then rescale our basis vectors to satisfy (3.66). The hard part, we would expect, is finding a coordinate system $x^{\hat{\mu}}$ for which the basis vectors $\{\hat{e}_{(\hat{\mu})}\}$ comprise the coordinate basis, $\hat{e}_{(\hat{\mu})} = \partial_{\hat{\mu}}$, and such that the first partial derivatives of $g_{\hat{\mu}\hat{\nu}}$ vanish. In fact, however, the exponential map achieves this automatically. For any point q sufficiently close to p, there is a unique geodesic path connecting p to q, and a unique parameterization λ such that $\lambda(p) = 0$ and $\lambda(q) = 1$. At p the tangent vector k to this geodesic can be written as a linear combination of our basis vectors, $k = k^{\hat{\mu}}\hat{e}_{(\hat{\mu})}$. We define the sought-after coordinates $x^{\hat{\mu}}$ simply to be these components: $x^{\hat{\mu}}(q) = k^{\hat{\mu}}$. In other words, we have defined the coordinates $x^{\hat{\mu}}(q)$ to be the components (with respect to our normalized basis $\{\hat{e}_{(\hat{\mu})}\}$) of the tangent vector k that gets mapped to q by $\exp_p$. Coordinates constructed in this way are known as **Riemann normal coordinates** at p.

We still need to verify that these Riemann normal coordinates satisfy $\partial_{\hat{\sigma}} g_{\hat{\mu}\hat{\nu}}(p) = 0$. Note that a ray in the tangent space (a parameterized set of vectors of the form $\lambda k^{\hat{\mu}}$, for some fixed vector $k^{\hat{\mu}}$) gets mapped to a geodesic by the exponential map. Therefore, in Riemann normal coordinates, a curve $x^{\hat{\mu}}(\lambda)$ of the form

$$x^{\hat{\mu}}(\lambda) = \lambda k^{\hat{\mu}} \tag{3.67}$$

will solve the geodesic equation. Indeed, *any* geodesic through p may be expressed this way, for some appropriate vector $k^{\hat{\mu}}$. We therefore have

$$\frac{d^2 x^{\hat{\mu}}}{d\lambda^2} = 0 \tag{3.68}$$

along any geodesic through p in this coordinate system. But, by the geodesic equation, we also have

$$\frac{d^2x^{\hat{\mu}}}{d\lambda^2}(p) = -\Gamma^{\hat{\mu}}_{\hat{\rho}\hat{\sigma}}(p)k^{\hat{\rho}}k^{\hat{\sigma}}, \tag{3.69}$$

where $k^{\hat{\rho}} = (dx^{\hat{\rho}}/d\lambda)(p)$. Since this holds for arbitrary $k^{\hat{\rho}}$, we conclude that

$$\Gamma^{\hat{\mu}}_{\hat{\rho}\hat{\sigma}}(p) = 0. \tag{3.70}$$

Now apply metric compatibility:

$$\begin{aligned} 0 &= \nabla_{\hat{\sigma}} g_{\hat{\mu}\hat{\nu}} \\ &= \partial_{\hat{\sigma}} g_{\hat{\mu}\hat{\nu}} - \Gamma^{\hat{\lambda}}_{\hat{\sigma}\hat{\mu}} g_{\hat{\lambda}\hat{\nu}} - \Gamma^{\hat{\lambda}}_{\hat{\sigma}\hat{\nu}} g_{\hat{\mu}\hat{\lambda}} \\ &= \partial_{\hat{\sigma}} g_{\hat{\mu}\hat{\nu}}, \end{aligned} \tag{3.71}$$

where all quantities are evaluated at p. We see that Riemann normal coordinates provide a realization of the locally inertial coordinates discussed in Section 2.5. They are not unique; there are an infinite number of non-Riemann-normal coordinate systems in which $g_{\hat{\mu}\hat{\nu}}(p) = \eta_{\hat{\mu}\hat{\nu}}$ and $\partial_{\hat{\sigma}} g_{\hat{\mu}\hat{\nu}}(p) = 0$, but in an expansion around p they will differ from the Riemann normal coordinates only at third order in $x^{\hat{\mu}}$.

3.5 ■ THE EXPANDING UNIVERSE REVISITED

Let's put some of the technology we have developed to work in understanding a simple metric. Recall the expanding-universe metric we studied in Chapter 2,

$$\begin{aligned} ds^2 &= -\mathrm{d}t^2 + a^2(t)[\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2] \\ &= -\mathrm{d}t^2 + a^2(t)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j. \end{aligned} \tag{3.72}$$

This metric describes a universe consisting of flat spatial sections expanding as a function of time, with the relative distance between particles at fixed spatial coordinates growing proportionally to the scale factor $a(t)$.

Faced with a metric, the first thing we do is to calculate the Christoffel symbols. As mentioned at the end of Section 3.3, the easiest technique for doing so is actually to vary explicitly an integral of the form (3.49). Plugging in the metric under consideration, we have

$$I = \frac{1}{2}\int \left[-\left(\frac{dt}{d\tau}\right)^2 + a^2(t)\delta_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau} \right] d\tau. \tag{3.73}$$

The technique is to consider variations $x^\mu \to x^\mu + \delta x^\mu$ and demand that δI vanish. We get n equations on an n-dimensional manifold (in this case $n = 4$), one for each μ; each equation corresponds to a component of the geodesic equation

(3.44). In the equation derived by varying with respect to x^μ, the coefficient of $(dx^\rho/d\tau)(dx^\sigma/d\tau)$ will be $\Gamma^\mu_{\rho\sigma}$.

For the metric (3.72), we need to consider separately variations with respect to $x^0 = t$ and one of the x^i's (it doesn't matter which one, since the results for each spacelike direction will be equivalent). Let's start with $t \to t + \delta t$. The nontrivial time dependence comes from the scale factor, for which, to first order,

$$a(t + \delta t) = a(t) + \dot{a}\delta t, \tag{3.74}$$

where $\dot{a} = da/dt$. We therefore have

$$\begin{aligned}\delta I &= \frac{1}{2}\int \left[-2\frac{dt}{d\tau}\frac{d(\delta t)}{d\tau} + 2a\dot{a}\delta_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}\delta t\right] d\tau \\ &= \int \left[\frac{d^2t}{d\tau^2} + a\dot{a}\delta_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}\right]\delta t\, d\tau,\end{aligned} \tag{3.75}$$

where we have dropped a boundary term after integrating by parts (as always). Setting the coefficient of δt equal to zero implies

$$\frac{d^2t}{d\tau^2} + a\dot{a}\delta_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau} = 0, \tag{3.76}$$

which is supposed to be equivalent to the geodesic equation (with $\mu = 0$)

$$\frac{d^2x^0}{d\tau^2} + \Gamma^0_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = 0. \tag{3.77}$$

Comparison of these two equations implies

$$\begin{aligned}\Gamma^0_{00} &= 0, \\ \Gamma^0_{i0} = \Gamma^0_{0i} &= 0, \\ \Gamma^0_{ij} &= a\dot{a}\,\delta_{ij}.\end{aligned} \tag{3.78}$$

We can repeat this procedure for a spatial coordinate, $x^i \to x^i + \delta x^i$. The variation is then

$$\begin{aligned}\delta I &= \frac{1}{2}\int a^2\left(2\delta_{ij}\frac{dx^i}{d\tau}\frac{d(\delta x^j)}{d\tau}\right) d\tau \\ &= -\int\left(a^2\frac{d^2x^i}{d\tau^2} + 2a\frac{da}{d\tau}\frac{dx^i}{d\tau}\right)\delta_{ij}\delta x^j\, d\tau.\end{aligned} \tag{3.79}$$

We can express $da/d\tau$ in terms of $\dot{a}$ by using the chain rule,

$$\frac{da}{d\tau} = \dot{a}\frac{dt}{d\tau}. \tag{3.80}$$

Then setting the coefficient of δx^j equal to zero in (3.79) implies

$$\frac{d^2x^i}{d\tau^2} + 2\frac{\dot{a}}{a}\frac{dt}{d\tau}\frac{dx^i}{d\tau} = 0. \tag{3.81}$$

Comparing to the geodesic equation, we find that the Christoffel symbols must satisfy

$$\Gamma^i_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = 2\frac{\dot{a}}{a}\frac{dt}{d\tau}\frac{dx^i}{d\tau}. \tag{3.82}$$

The Christoffel symbols are therefore given by

$$\begin{aligned} \Gamma^i_{00} &= 0 \\ \Gamma^i_{j0} = \Gamma^i_{0j} &= \frac{\dot{a}}{a}\delta^i_j \\ \Gamma^i_{jk} &= 0. \end{aligned} \tag{3.83}$$

Together, (3.78) and (3.83) are all of the connection coefficients for the metric (3.72). These are, of course, necessary both for studying geodesics of the spacetime and for taking covariant derivatives; in fact, (3.76) and (3.81) together *are* the geodesic equation. Let's put this to work by solving for null geodesics, those followed by massless particles such as photons, for which we have to use λ rather than τ as a parameter. Without loss of generality we can consider paths along the x-direction, for which $x^\mu(\lambda) = \{t(\lambda), x(\lambda), 0, 0\}$. It is trivial to solve for null paths of this sort, using $ds^2 = 0$. We have

$$0 = -\mathrm{d}t^2 + a^2(t)\mathrm{d}x^2, \tag{3.84}$$

which implies

$$\frac{dx}{d\lambda} = \frac{1}{a}\frac{dt}{d\lambda}. \tag{3.85}$$

In Section 2.6 we solved this for $a = t^q$, but here we will remain more general. Also, we have chosen to consider paths moving in the positive x-direction, which determines the sign of $dx/d\lambda$. We must distinguish, however, between "null paths" and "null geodesics": the latter are a much more restrictive class, and to show that these paths are geodesics, we need to solve for the coordinates t and x in terms of the parameter λ.

Let's solve for $dt/d\lambda$, which will turn out to be the quantity in which we are most interested. Plugging the null condition (3.85) into the $\mu = 0$ component of the geodesic equation (3.76), and remembering to replace $\tau \to \lambda$, we get

$$\frac{d^2t}{d\lambda^2} + \frac{\dot{a}}{a}\left(\frac{dt}{d\lambda}\right)^2 = 0. \tag{3.86}$$

It is straightforward to verify that this is solved by

$$\frac{dt}{d\lambda} = \frac{\omega_0}{a}, \tag{3.87}$$

where ω_0 is a constant. For a given $a(t)$, this could be instantly integrated to yield $t(\lambda)$. But more interesting is to consider the energy E of the photon as it would be measured by a comoving observer (one at fixed spatial coordinates), who would have four-velocity

$$U^\mu = (1, 0, 0, 0). \tag{3.88}$$

Don't get tricked into thinking that the timelike component of the four-velocity of a particle at rest will always equal unity; we need to satisfy the normalization condition $g_{\mu\nu}U^\mu U^\nu = -1$, which in the rest frame ($U^i = 0$) implies $U^0 = \sqrt{-g_{00}}$. According to (3.63), and using $p^\mu = dx^\mu/d\lambda$, we have

$$\begin{aligned} E &= -p_\mu U^\mu \\ &= -g_{00}\frac{dx^0}{d\lambda}U^0 \\ &= \frac{\omega_0}{a}. \end{aligned} \tag{3.89}$$

We see why the notation ω_0 was chosen for the constant of proportionality in (3.87): ω_0 is simply the frequency of the photon when $a = 1$. Recall that $E = \hbar\omega$, and we are using units in which $\hbar = 1$.

We have uncovered a profound phenomenon: the **cosmological redshift**. A photon emitted with energy E_1 at scale factor a_1 and observed with energy E_2 at scale factor a_2 will have

$$\frac{E_2}{E_1} = \frac{a_1}{a_2}. \tag{3.90}$$

This is called a "redshift" because the wavelength of the photon is inversely proportional to the frequency, and in an expanding universe the wavelength therefore grows with time. As a practical matter this provides an easy way to measure the change in the scale factor between us and distant galaxies, and also serves as a proxy for the distance: since the universe has been monotonically expanding, a greater redshift implies a greater distance. In conventional notation, the amount of redshift is denoted by

$$z = \frac{\omega_1 - \omega_2}{\omega_2} = \frac{a_2}{a_1} - 1, \tag{3.91}$$

so that z vanishes if there has been no expansion, for instance, if the emitter and observer are so close that there hasn't been enough time for the universe to appreciably expand.

As mentioned in Section 3.3, the cosmological redshift is *not* a Doppler shift (despite the understandable temptation to refer to the "velocity" of receding galaxies). Now we can understand this statement quantitatively. You might imagine that, as far as the behavior of emitted photons is concerned, there is little difference between two galaxies physically moving apart in a flat spacetime and two galaxies at fixed comoving coordinates in an expanding spacetime. But let's consider a specific (unrealistic, but educational) example. Start with flat spacetime, and imagine that our two galaxies are initially not moving apart, but are at rest in some globally inertial coordinate system. One emits a photon toward the other; while the photon is traveling, we quickly move the two galaxies apart until they are twice their original separation, then leave them stationary at that distance; and then the photon is absorbed by the second galaxy. Clearly there will be no Doppler shift, since the galaxies were at rest both at emission and absorption. Now consider the analogous phenomenon in an expanding spacetime, with the galaxies stuck at fixed comoving coordinates. We begin with the scale factor constant (the universe is not expanding). One galaxy emits a photon, and we imagine that during the photon's journey the universe starts expanding until the scale factor is twice its original size, and then stops expanding before the photon is absorbed. In this case there certainly will be a redshift, despite the fact that there was no "relative motion" (an ill-defined concept in any case) at either absorption or emission; the photon's wavelength will have doubled as the scale factor doubled, so we observe a redshift $z = 1$. This demonstrates the conceptual distinction between the cosmological redshift and the conventional Doppler effect.

Beyond the geodesic equation, covariant derivatives will play a role in generalizing laws of physics from the flat spacetime of special relativity to the curved geometry of general relativity. As we will discuss in more detail in the next chapter, a simple rule of thumb is simply to replace all partial derivatives by covariant derivatives, and all appearances of the flat spacetime metric $\eta_{\mu\nu}$ by the curved metric $g_{\mu\nu}$. For example, the energy-momentum conservation equation of special relativity, $\partial_\mu T^{\mu\nu} = 0$, where $T^{\mu\nu}$ is the energy-momentum tensor, becomes

$$\nabla_\mu T^{\mu\nu} = 0. \tag{3.92}$$

In cosmology, we typically model the matter filling the universe as a perfect fluid; the corresponding energy-momentum tensor comes from generalizing (1.114) to curved spacetime,

$$T^{\mu\nu} = (\rho + p)U^\mu U^\nu + pg^{\mu\nu}. \tag{3.93}$$

Recall that ρ is the energy density, p is the pressure, and U^μ is the four-velocity of the fluid. For the metric (3.72) the components of the inverse metric are

$$g^{\mu\nu} = \begin{pmatrix} -1 & & & \\ & a^{-2} & & \\ & & a^{-2} & \\ & & & a^{-2} \end{pmatrix}. \tag{3.94}$$

We can take the fluid to be in its rest frame in these coordinates, so that the components of the four-velocity are $U^\mu = (1, 0, 0, 0)$. In fact the fluid would have to be in its rest frame for this particular metric to solve Einstein's equation, as we will later see. The energy-momentum tensor therefore takes the form

$$T^{\mu\nu} = \begin{pmatrix} \rho & & & \\ & a^{-2}p & & \\ & & a^{-2}p & \\ & & & a^{-2}p \end{pmatrix}. \tag{3.95}$$

Note that these components are specific to the metric (3.72), and will generally look different for other metrics.

Let's see what the energy-momentum conservation equation $\nabla_\mu T^{\mu\nu} = 0$ implies for a perfect fluid in an expanding universe. The rule for covariant derivatives (3.17) implies

$$\nabla_\mu T^{\mu\nu} = \partial_\mu T^{\mu\nu} + \Gamma^\mu_{\mu\lambda} T^{\lambda\nu} + \Gamma^\nu_{\mu\lambda} T^{\mu\lambda} = 0. \tag{3.96}$$

This equation has four components, one for each μ, although the three $\mu = i \in \{1, 2, 3\}$ are equivalent. Let's first look at the $\nu = 0$ component, piece by piece. The first term is straightforward,

$$\partial_\mu T^{\mu 0} = \partial_0 T^{00} = \dot{\rho}. \tag{3.97}$$

The second term is

$$\Gamma^\mu_{\mu\lambda} T^{\lambda 0} = \Gamma^\mu_{\mu 0} T^{00} = 3\frac{\dot{a}}{a}\rho, \tag{3.98}$$

and the third term is

$$\Gamma^0_{\mu\lambda} T^{\mu\lambda} = \Gamma^0_{00} T^{00} + \Gamma^0_{11} T^{11} + \Gamma^0_{22} T^{22} + \Gamma^0_{33} T^{33} = 3\frac{\dot{a}}{a}p. \tag{3.99}$$

In each of these sets of equations, we have first invoked the fact that $T^{\mu\nu}$ is diagonal, and then used the explicit formulae for the energy-momentum tensor and the connection coefficients in this metric. All together, then, we find

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + p). \tag{3.100}$$

Now let's look at one of the spatial components, choosing $\nu = 1$ for definiteness. Once again working piece by piece, we have for the first term in (3.96),

$$\partial_\mu T^{\mu 1} = \partial_1 T^{11} = a^{-2}\partial_x p. \tag{3.101}$$

The second and third terms are

$$\Gamma^\mu_{\mu\lambda} T^{\lambda 1} = \Gamma^\mu_{\mu 1} T^{11} = 0, \tag{3.102}$$

and

$$\Gamma^1_{\mu\lambda}T^{\mu\lambda} = \Gamma^1_{00}T^{00} + \Gamma^1_{11}T^{11} + \Gamma^1_{22}T^{22} + \Gamma^1_{33}T^{33} = 0. \tag{3.103}$$

Equivalent results will hold for $\nu = 2$ and $\nu = 3$. So the spatial components of the energy-momentum conservation equation simply amount to

$$\partial_i p = 0. \tag{3.104}$$

It is illuminating to compare these results to those we would obtain in Minkowski spacetime, which can be found by simply setting $a = 1$, $\dot{a} = 0$. The pressure-gradient equation (3.104) is unaffected, so there is no effect of curvature on the spatial components: for a fluid that is motionless as measured by a comoving observer, the pressure must be constant throughout space. For the timelike component, on the other hand, the expansion of the universe introduces a nonzero right-hand side to (3.100). To understand the consequences of this new feature, let us consider equations of state of the form

$$p = w\rho, \tag{3.105}$$

where w is some constant. Then (3.100) becomes

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a}, \tag{3.106}$$

which can be solved to yield

$$\rho \propto a^{-3(1+w)}. \tag{3.107}$$

In Chapter 1 we mentioned three kinds of perfect fluid with equations of state of the form (3.105): dust, with $w = 0$; radiation, with $w = \frac{1}{3}$; and vacuum, with $w = -1$. A set of nonrelativistic, noninteracting particles behaves like dust; a set of photons or other massless particles behaves like radiation; and a nonzero constant energy density throughout spacetime acts like vacuum. From (3.107) we see that the equation of state determines how the energy density evolves as the universe expands:

$$\begin{aligned} &\text{matter} && p = 0 && \rho \propto a^{-3} \\ &\text{radiation} && p = \tfrac{1}{3}\rho && \rho \propto a^{-4} \\ &\text{vacuum} && p = -\rho && \rho = \text{constant}. \end{aligned} \tag{3.108}$$

We will explore these behaviors more thoroughly in Chapter 8; for right now let's simply note that they make sense. For dust, the energy density comes from the rest mass of each particle; if all the particles have mass m, the energy density is simply $\rho = nm$, where n is the number density. Since the number density goes down as a^{-3} (the physical volume of a comoving region goes up, while the

total number of particles stays constant), while the masses remain unchanged, we expect that the energy density obeys $\rho \propto a^{-3}$. For radiation, meanwhile, the energy of each particle (such as a photon) redshifts away as a^{-1} as the universe expands; since the number density still obeys $n \propto a^{-3}$, we expect that $\rho \propto a^{-4}$. Finally, the vacuum energy density is an intrinsic and unchanging amount of energy in any physical volume; it doesn't redshift at all as the universe expands, so we get $\rho =$ constant.

This example brings to life the differences between flat and curved spacetimes. For example, consider what we might be tempted to call the "energy," the integral over space of the energy density: $E = \int \rho a^3 \, d^3x$, where the boundaries are at fixed comoving coordinates, so the region expands along with the universe, and the factor of a^3 comes from the square root of the determinant of the spatial metric $a^2\delta_{ij}$. This number is clearly not conserved in general. For dust, since $\rho \propto a^{-3}$, E remains constant as the universe expands; but for radiation it decreases, and for vacuum energy it increases. This is upsetting, since conservation of energy is one of the more cherished principles of physics. What has happened? One way of thinking about this is from the viewpoint of Noether's theorem, which states that every symmetry implies a conserved quantity. Energy is the conserved quantity that derives from invariance under time translations. Clearly, in an expanding universe, the energy-momentum tensor is defined on a background that is changing with time; therefore there is no reason to believe that the energy should be conserved. ("There is no timelike Killing vector," in the language to be introduced in Section 3.8.) Nevertheless, we continue to refer to $\nabla_\mu T^{\mu\nu} = 0$ as the energy-momentum conservation equation. It conveys the idea that there is a definite law obeyed by the energy-momentum tensor, even if there is no integral corresponding to a conserved energy. The transition from flat to curved spacetime induces the additional Christoffel-symbol terms in (3.96); these terms serve, roughly speaking, to allow transfer of energy between the matter fields (comprising $T^{\mu\nu}$) and the gravitational field. This notion is not very formal, however, and you shouldn't push it too far—it turns out to be difficult to associate a local energy density to the gravitational field, although it is possible in certain circumstances.

Of course there is also a notion of time-translation invariance that refers not to the background spacetime, but to the theory itself (that is, to the equations that define the theory rather than a specific solution to them). We haven't yet developed the dynamical equations of general relativity, but they will turn out to be invariant under time translations, as well as under any other sort of coordinate transformations, as indeed they must be. This general coordinate invariance leads to a set of constraints on allowed configurations of the theory, and generally requires a more subtle analysis.

In the end, you should come to accept that there is a profound difference between flat and curved spacetimes, and some of our favorite notions from flat-spacetime physics will be seriously modified in this more general context. This is not a sign of any flaw in general relativity, but a natural consequence of discarding the rigid spacetime geometry we learn to take for granted.

3.6 ■ THE RIEMANN CURVATURE TENSOR

Having set up the machinery of covariant derivatives and parallel transport, we are at last prepared to discuss curvature proper. The curvature is quantified by the Riemann tensor, which is derived from the connection. The idea behind this measure of curvature is that we know what we mean by "flatness" of a connection—the conventional (and usually implicit) Christoffel connection associated with a Euclidean or Minkowskian metric has a number of properties that can be thought of as different manifestations of flatness. These include the fact that parallel transport around a closed loop leaves a vector unchanged, that covariant derivatives of tensors commute, and that initially parallel geodesics remain parallel. As we shall see, the Riemann tensor arises when we study how any of these properties are altered in more general contexts.

We have already argued, using the two-sphere as an example, that parallel transport of a vector around a closed loop in a curved space will lead to a transformation of the vector. The resulting transformation depends on the total curvature enclosed by the loop; it would be more useful to have a local description of the curvature at each point, which is what the Riemann tensor is supposed to provide. One conventional way to introduce the Riemann tensor, therefore, is to consider parallel transport around an infinitesimal loop. We are not going to do that here, but take a more direct route. Nevertheless, even without working through the details, it is possible to see what form the answer should take. Since spacetime looks flat in sufficiently small regions, our loop will be specified by two (infinitesimal) vectors A^μ and B^ν. We imagine parallel transporting a vector V^μ by first moving it in the direction of A^μ, then along B^ν, then backward along A^μ and B^ν to return to the starting point, as shown in Figure 3.5. We know the action of parallel transport is independent of coordinates, so there should be some tensor that tells us how the vector changes when it comes back to its starting point; it will be a linear transformation on a vector, and therefore involve one upper and one lower index. But it will also depend on the two vectors A and B that define the loop; therefore there should be two additional lower indices to contract with A^μ and B^ν. Furthermore, the tensor should be antisymmetric in these two indices, since interchanging the vectors corresponds to traversing the loop in the opposite direction, and should give the inverse of the original answer. This is consistent with the fact that the transformation should vanish if A and B are the same vector. We therefore expect that the expression for the change δV^ρ experienced by this vector when parallel transported around the loop should be of the form

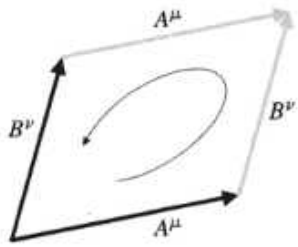


FIGURE 3.5 An infinitesimal loop defined by two vectors A^μ and B^ν.

$$\delta V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma A^\mu B^\nu, \tag{3.109}$$

where $R^\rho{}_{\sigma\mu\nu}$ is a $(1,3)$ tensor known as the **Riemann tensor** (or simply curvature tensor). It is antisymmetric in the last two indices:

$$R^\rho{}_{\sigma\mu\nu} = -R^\rho{}_{\sigma\nu\mu}. \tag{3.110}$$

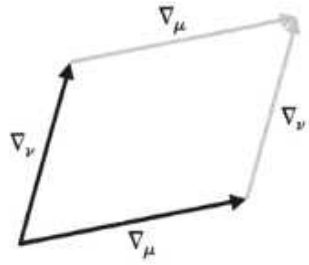


FIGURE 3.6 The commutator of two covariant derivatives.

Of course, if (3.109) is taken as a definition of the Riemann tensor, a convention needs to be chosen for the ordering of the indices. There is no agreement at all on what this convention should be, so be careful.

Knowing what we do about parallel transport, we could very carefully perform the necessary manipulations to see what happens to the vector under this operation, and the result would be a formula for the curvature tensor in terms of the connection coefficients. It is much quicker, however, to consider a related operation, the commutator of two covariant derivatives. The relationship between this and parallel transport around a loop should be evident; the covariant derivative of a tensor in a certain direction measures how much the tensor changes relative to what it would have been if it had been parallel transported, since the covariant derivative of a tensor in a direction along which it is parallel transported is zero. The commutator of two covariant derivatives, then, measures the difference between parallel transporting the tensor first one way and then the other, versus the opposite ordering, as shown in Figure 3.6.

The actual computation is very straightforward. Considering a vector field V^ρ, we take

$$\begin{aligned}
[\nabla_\mu, \nabla_\nu]V^\rho &= \nabla_\mu\nabla_\nu V^\rho - \nabla_\nu\nabla_\mu V^\rho \\
&= \partial_\mu(\nabla_\nu V^\rho) - \Gamma^\lambda_{\mu\nu}\nabla_\lambda V^\rho + \Gamma^\rho_{\mu\sigma}\nabla_\nu V^\sigma - (\mu \leftrightarrow \nu) \\
&= \partial_\mu\partial_\nu V^\rho + (\partial_\mu\Gamma^\rho_{\nu\sigma})V^\sigma + \Gamma^\rho_{\nu\sigma}\partial_\mu V^\sigma - \Gamma^\lambda_{\mu\nu}\partial_\lambda V^\rho - \Gamma^\lambda_{\mu\nu}\Gamma^\rho_{\lambda\sigma}V^\sigma \\
&\quad + \Gamma^\rho_{\mu\sigma}\partial_\nu V^\sigma + \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\nu\lambda}V^\lambda - (\mu \leftrightarrow \nu) \\
&= (\partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma})V^\sigma - 2\Gamma^\lambda_{[\mu\nu]}\nabla_\lambda V^\rho.
\end{aligned} \tag{3.111}$$

In the last step we have relabeled some dummy indices and eliminated some terms that cancel when antisymmetrized. We recognize that the antisymmetrized connection coefficients in the last term are simply one-half times the torsion tensor, and that the left hand side is manifestly a tensor; therefore the expression in parentheses must be a tensor itself. We write

$$[\nabla_\mu, \nabla_\nu]V^\rho = R^\rho{}_{\sigma\mu\nu}V^\sigma - T^\lambda{}_{\mu\nu}\nabla_\lambda V^\rho, \tag{3.112}$$

where the Riemann tensor is identified as

$$\boxed{R^\rho{}_{\sigma\mu\nu} = \partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}.} \tag{3.113}$$

Notice a number of things about the derivation of this expression:

- Of course we have not demonstrated that (3.113) is actually the same tensor that appeared in (3.109), but in fact it's true. You are asked to show this in the Exercises.
- It is perhaps surprising that the commutator $[\nabla_\mu, \nabla_\nu]$, which appears to be a differential operator, has an action on vector fields that (in the absence of

torsion, at any rate) is a simple multiplicative transformation. The Riemann tensor measures that part of the commutator of covariant derivatives that is proportional to the vector field, while the torsion tensor measures the part that is proportional to the covariant derivative of the vector field; the second derivative doesn't enter at all.

- Notice that the expression (3.113) is constructed from nontensorial elements; you can check that the transformation laws all work out to make this particular combination a legitimate tensor.
- The antisymmetry of $R^{\rho}{}_{\sigma\mu\nu}$ in its last two indices is immediate from this formula and its derivation.
- We constructed the curvature tensor completely from the connection (no mention of the metric was made). We were sufficiently careful that the above expression is true for any connection, whether or not it is metric compatible or torsion free.
- Using what are by now our usual methods, the action of $[\nabla_\rho, \nabla_\sigma]$ can be computed on a tensor of arbitrary rank. The answer is

$$\begin{aligned}
[\nabla_\rho, \nabla_\sigma] X^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} &= -T^{\lambda}{}_{\rho\sigma} \nabla_\lambda X^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} \\
&\quad + R^{\mu_1}{}_{\lambda\rho\sigma} X^{\lambda \mu_2 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} + R^{\mu_2}{}_{\lambda\rho\sigma} X^{\mu_1 \lambda \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} + \cdots \\
&\quad - R^{\lambda}{}_{\nu_1 \rho\sigma} X^{\mu_1 \cdots \mu_k}{}_{\lambda \nu_2 \cdots \nu_l} - R^{\lambda}{}_{\nu_2 \rho\sigma} X^{\mu_1 \cdots \mu_k}{}_{\nu_1 \lambda \cdots \nu_l} - \cdots . \qquad (3.114)
\end{aligned}$$

Both the torsion tensor and the Riemann tensor, thought of as multilinear maps, have elegant expressions in terms of the vector-field commutator. Thinking of the torsion as a map from two vector fields to a third vector field, we have

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y], \qquad (3.115)$$

and thinking of the Riemann tensor as a map from three vector fields to a fourth one, we have (in funny-looking but standard notation)

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \qquad (3.116)$$

In these expressions, the notation ∇_X refers to the covariant derivative along the vector field X; in components, $\nabla_X = X^\mu \nabla_\mu$. So, for example, (3.116) is equivalent to

$$\begin{aligned}
R^{\rho}{}_{\sigma\mu\nu} X^\mu Y^\nu Z^\sigma &= X^\lambda \nabla_\lambda (Y^\eta \nabla_\eta Z^\rho) - Y^\lambda \nabla_\lambda (X^\eta \nabla_\eta Z^\rho) \\
&\quad - (X^\lambda \partial_\lambda Y^\eta - Y^\lambda \partial_\lambda X^\eta) \nabla_\eta Z^\rho , \qquad (3.117)
\end{aligned}$$

which you can check is equivalent to (3.113). Note that the two vectors X and Y in (3.116) correspond to the last two indices in the component form of the Riemann

tensor. The last term in (3.116), involving the commutator $[X, Y]$, vanishes when X and Y are taken to be the coordinate basis vector fields (since $[\partial_\mu, \partial_\nu] = 0$), which is why this term did not arise when we originally took the commutator of two covariant derivatives. We will not use this notation extensively, but you might see it in the literature, so you should be able to decode it.

Having defined the curvature tensor as something that characterizes the connection, let us now admit that in GR we are most concerned with the Christoffel connection. In this case the connection is derived from the metric, and the associated curvature may be thought of as that of the metric itself. This identification allows us to finally make sense of our informal notion that spaces for which the metric looks Euclidean or Minkowskian are flat. In fact it works both ways:

- If a coordinate system exists in which the components of the metric are constant, the Riemann tensor will vanish.
- If the Riemann tensor vanishes, we can always construct a coordinate system in which the metric components are constant.

Technically, these statements should be restricted to a region of the manifold that is simply-connected (all loops in the region can be smoothly deformed to a point without leaving the region); we will implicitly assume this condition below.

The first of these is easy to show. If we are in some coordinate system such that $\partial_\sigma g_{\mu\nu} = 0$ everywhere, not just at a point, then $\Gamma^\rho_{\mu\nu} = 0$ and $\partial_\sigma \Gamma^\rho_{\mu\nu} = 0$; thus $R^\rho{}_{\sigma\mu\nu} = 0$ by (3.113). But this is a tensor equation, and if it is true in one coordinate system it must be true in any coordinate system. Therefore, the statement that the Riemann tensor vanishes is a necessary condition for it to be possible to find coordinates in which the components of $g_{\mu\nu}$ are constant everywhere.

The second claim, that $R^\rho{}_{\sigma\mu\nu} = 0$ everywhere implies we can find a coordinate system in which the metric components are constant everywhere, is harder to prove (but not very hard). Consider as a warm-up a one-form $\omega = \omega_\mu dx^\mu$, defined at some point p. For any path $x^\mu(\lambda)$ that includes p, we can construct a unique one-form field along the path by demanding that ω_μ be parallel-transported:

$$\frac{dx^\mu}{d\lambda}\nabla_\mu \omega_\nu = 0. \tag{3.118}$$

In general, if we performed this procedure for distinct paths that started at p and passed through some other point q, the value of ω_μ at q would depend on the path. However, if the Riemann tensor vanishes everywhere, the parallel-transport will be path-independent, and we can define a unique one-form field throughout the manifold. Therefore (3.118) must be true for arbitrary $dx^\mu/d\lambda$; this can only be true if ω_μ is covariantly constant:

$$\nabla_\mu \omega_\nu = 0. \tag{3.119}$$

On an arbitrary manifold there will be no solutions to this equation; it is only possible here because we are assuming that the curvature vanishes. We can take

the antisymmetric part of (3.119), and from (3.36) we know that this is just the exterior derivative:

$$\nabla_{[\mu}\omega_{\nu]} = \partial_{[\mu}\omega_{\nu]} = 0, \tag{3.120}$$

or, in index-free notation,

$$\mathrm{d}\omega = 0. \tag{3.121}$$

In other words, ω is closed. It is also exact (there exists a scalar function α such that $\omega = \mathrm{d}\alpha$), since we have restricted the topology of the region in which we are working. In components we have

$$\omega_\mu = \partial_\mu \alpha. \tag{3.122}$$

There is nothing special about the one-form ω, so we can repeat this procedure with a set of one-forms $\hat{\theta}^{(a)}$, where $a \in \{1 \ldots n\}$ on an n-dimensional manifold. We may choose our one-forms to comprise a normalized basis for the dual space T_p^*, such that the components of the metric with respect to this basis are those of the canonical form; in other words,

$$ds^2(p) = \eta_{ab}\,\hat{\theta}^{(a)} \otimes \hat{\theta}^{(b)}. \tag{3.123}$$

Here we use η_{ab} in a generalized sense, as a matrix with either $+1$ or -1 for each diagonal element and zeroes elsewhere. The actual arrangement of the $+1$'s and -1's depends on the canonical form of the metric, but is irrelevant for the present argument. Now let us parallel transport the entire set of basis forms all over the manifold; the vanishing of the Riemann tensor ensures that the result will be independent of the path taken. Since the metric is always automatically parallel-transported with respect to a metric-compatible connection, the metric components will remain unchanged,

$$ds^2(\text{anywhere}) = \eta_{ab}\,\hat{\theta}^{(a)} \otimes \hat{\theta}^{(b)}. \tag{3.124}$$

We therefore have specified a set of one-form fields, which everywhere define a basis in which the metric components are constant. This is completely unimpressive; it can be done on any manifold, regardless of what the curvature is. What we would like to show is that this is a *coordinate* basis (which will only be possible if the curvature vanishes). However, by the same arguments that led to (3.122), we know that all of the $\hat{\theta}^{(a)}$'s are exact forms, so that there exists a set of functions y^a such that the one-form fields are their gradients,

$$\hat{\theta}^{(a)} = \mathrm{d}y^a. \tag{3.125}$$

These n functions are precisely the sought-after coordinates; all over the manifold the metric takes the form

$$ds^2 = \eta_{ab}\,\mathrm{d}y^a \mathrm{d}y^b. \tag{3.126}$$

At this point you are welcome to switch from using a, b as indices to μ, ν if you prefer.

We have thus verified that the Riemann tensor provides us with an answer to the question of whether some horrible-looking metric is secretly that of flat space in a perverse coordinate system. If we calculate the Riemann tensor of such a metric and find that it vanishes, we know that the metric is flat; if it doesn't vanish, there is curvature.

3.7 ■ PROPERTIES OF THE RIEMANN TENSOR

The Riemann tensor, with four indices, naively has n^4 independent components in an n-dimensional space. In fact the antisymmetry property (3.110) means that there are only $n(n-1)/2$ independent values these last two indices can take on, leaving us with $n^3(n-1)/2$ independent components. When we consider the Christoffel connection, however, a number of other symmetries reduce the number of independent components further. Let's consider these now.

The simplest way to derive these additional symmetries is to examine the Riemann tensor with all lower indices,

$$R_{\rho\sigma\mu\nu} = g_{\rho\lambda} R^{\lambda}{}_{\sigma\mu\nu}. \tag{3.127}$$

Let us further consider the components of this tensor in locally inertial coordinates $x^{\hat{\mu}}$ established at a point p. Then the Christoffel symbols themselves will vanish, although their derivatives will not. We therefore have

$$\begin{aligned} R_{\hat{\rho}\hat{\sigma}\hat{\mu}\hat{\nu}}(p) &= g_{\hat{\rho}\hat{\lambda}}(\partial_{\hat{\mu}}\Gamma^{\hat{\lambda}}_{\hat{\nu}\hat{\sigma}} - \partial_{\hat{\nu}}\Gamma^{\hat{\lambda}}_{\hat{\mu}\hat{\sigma}}) \\ &= \tfrac{1}{2} g_{\hat{\rho}\hat{\lambda}} g^{\hat{\lambda}\hat{\tau}} (\partial_{\hat{\mu}}\partial_{\hat{\nu}} g_{\hat{\sigma}\hat{\tau}} + \partial_{\hat{\mu}}\partial_{\hat{\sigma}} g_{\hat{\tau}\hat{\nu}} - \partial_{\hat{\mu}}\partial_{\hat{\tau}} g_{\hat{\nu}\hat{\sigma}} - \partial_{\hat{\nu}}\partial_{\hat{\mu}} g_{\hat{\sigma}\hat{\tau}} \\ &\qquad - \partial_{\hat{\nu}}\partial_{\hat{\sigma}} g_{\hat{\tau}\hat{\mu}} + \partial_{\hat{\nu}}\partial_{\hat{\tau}} g_{\hat{\mu}\hat{\sigma}}) \\ &= \tfrac{1}{2}(\partial_{\hat{\mu}}\partial_{\hat{\sigma}} g_{\hat{\rho}\hat{\nu}} - \partial_{\hat{\mu}}\partial_{\hat{\rho}} g_{\hat{\nu}\hat{\sigma}} - \partial_{\hat{\nu}}\partial_{\hat{\sigma}} g_{\hat{\rho}\hat{\mu}} + \partial_{\hat{\nu}}\partial_{\hat{\rho}} g_{\hat{\mu}\hat{\sigma}}). \end{aligned} \tag{3.128}$$

In the first line we have used $\Gamma^{\hat{\tau}}_{\hat{\mu}\hat{\nu}}(p) = 0$, in the second line we have used $\partial_{\hat{\mu}} g^{\hat{\lambda}\hat{\tau}} = 0$ in Riemann normal coordinates, and the fact that partials commute in the third line. From this expression we can notice immediately three properties of $R_{\rho\sigma\mu\nu}$: it is antisymmetric in its first two indices,

$$\boxed{R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu},} \tag{3.129}$$

it is antisymmetric in its last two indices [which we already knew from (3.110)],

$$\boxed{R_{\rho\sigma\mu\nu} = -R_{\rho\sigma\nu\mu},} \tag{3.130}$$

and it is invariant under interchange of the first pair of indices with the second:

$$\boxed{R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}.} \tag{3.131}$$

With a little more work, which is left to your imagination, we can see that the sum of cyclic permutations of the last three indices vanishes:

$$\boxed{R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\mu} = 0.} \tag{3.132}$$

Given (3.130), it's easy to see that this last property is equivalent to the vanishing of the antisymmetric part of the last three indices:

$$\boxed{R_{\rho[\sigma\mu\nu]} = 0.} \tag{3.133}$$

All of these properties have been derived in a special coordinate system, but they are all tensor equations; therefore they will be true in any coordinates (so we haven't bothered with hats on the indices). Not all of them are independent; with some effort, you can show that (3.129), (3.130), and (3.133) together imply (3.131). The logical interdependence of the equations is usually less important than the fact that they are true.

Given these relationships between the different components of the Riemann tensor, how many independent quantities remain? Let's begin with the facts that $R_{\rho\sigma\mu\nu}$ is antisymmetric in the first two indices, antisymmetric in the last two indices, and symmetric under interchange of these two pairs. This means that we can think of it as a symmetric matrix $R_{[\rho\sigma][\mu\nu]}$, where the pairs $\rho\sigma$ and $\mu\nu$ are thought of as individual indices. An $m \times m$ symmetric matrix has $m(m+1)/2$ independent components, while an $n \times n$ antisymmetric matrix has $n(n-1)/2$ independent components. We therefore have

$$\tfrac{1}{2}\left[\tfrac{1}{2}n(n-1)\right]\left[\tfrac{1}{2}n(n-1)+1\right] = \tfrac{1}{8}(n^4 - 2n^3 + 3n^2 - 2n) \tag{3.134}$$

independent components. We still have to deal with the additional symmetry (3.133). An immediate consequence of (3.133) is that the totally antisymmetric part of the Riemann tensor vanishes,

$$R_{[\rho\sigma\mu\nu]} = 0. \tag{3.135}$$

In fact, this equation plus the other symmetries (3.129), (3.130), and (3.131), are enough to imply (3.133), as can be easily shown by expanding (3.135) and manipulating the resulting terms. Therefore imposing the additional constraint of (3.135) is equivalent to imposing (3.133), once the other symmetries have been accounted for. How many independent restrictions does this represent? Let us imagine decomposing

$$R_{\rho\sigma\mu\nu} = X_{\rho\sigma\mu\nu} + R_{[\rho\sigma\mu\nu]}. \tag{3.136}$$

It is easy to see that any totally antisymmetric 4-index tensor is automatically antisymmetric in its first and last indices, and symmetric under interchange of the two pairs. Therefore these properties are independent restrictions on $X_{\rho\sigma\mu\nu}$, unrelated to the requirement (3.135). Now a totally antisymmetric 4-index tensor has $n(n-1)(n-2)(n-3)/4!$ terms, and therefore (3.135) reduces the number of independent components by this amount. We are left with

$$\tfrac{1}{8}(n^4 - 2n^3 + 3n^2 - 2n) - \tfrac{1}{24}n(n-1)(n-2)(n-3) = \tfrac{1}{12}n^2(n^2-1) \tag{3.137}$$

independent components of the Riemann tensor.

In four dimensions, therefore, the Riemann tensor has 20 independent components. (In one dimension it has none.) These twenty functions are precisely the 20 degrees of freedom in the second derivatives of the metric that we could not set to zero by a clever choice of coordinates when we first discussed locally inertial coordinates in Chapter 2. This should reinforce your confidence that the Riemann tensor is an appropriate measure of curvature.

In addition to the algebraic symmetries of the Riemann tensor (which constrain the number of independent components at any point), it obeys a differential identity, which constrains its relative values at different points. Consider the covariant derivative of the Riemann tensor, evaluated in locally inertial coordinates:

$$\begin{aligned}\nabla_{\hat{\lambda}} R_{\hat{\rho}\hat{\sigma}\hat{\mu}\hat{\nu}} &= \partial_{\hat{\lambda}} R_{\hat{\rho}\hat{\sigma}\hat{\mu}\hat{\nu}} \\ &= \tfrac{1}{2}\partial_{\hat{\lambda}}(\partial_{\hat{\mu}}\partial_{\hat{\sigma}} g_{\hat{\rho}\hat{\nu}} - \partial_{\hat{\mu}}\partial_{\hat{\rho}} g_{\hat{\nu}\hat{\sigma}} - \partial_{\hat{\nu}}\partial_{\hat{\sigma}} g_{\hat{\rho}\hat{\mu}} + \partial_{\hat{\nu}}\partial_{\hat{\rho}} g_{\hat{\mu}\hat{\sigma}}).\end{aligned} \tag{3.138}$$

It may seem illegitimate to take the derivative of an expression that is only true at a point, but the terms we are neglecting are all proportional to $\partial_{\hat{\sigma}} g_{\hat{\mu}\hat{\nu}}$, and therefore vanish. We would like to consider the sum of cyclic permutations of the first three indices:

$$\begin{aligned}&\nabla_{\hat{\lambda}} R_{\hat{\rho}\hat{\sigma}\hat{\mu}\hat{\nu}} + \nabla_{\hat{\rho}} R_{\hat{\sigma}\hat{\lambda}\hat{\mu}\hat{\nu}} + \nabla_{\hat{\sigma}} R_{\hat{\lambda}\hat{\rho}\hat{\mu}\hat{\nu}} \\ &\quad = \tfrac{1}{2}(\partial_{\hat{\lambda}}\partial_{\hat{\mu}}\partial_{\hat{\sigma}} g_{\hat{\rho}\hat{\nu}} - \partial_{\hat{\lambda}}\partial_{\hat{\mu}}\partial_{\hat{\rho}} g_{\hat{\nu}\hat{\sigma}} - \partial_{\hat{\lambda}}\partial_{\hat{\nu}}\partial_{\sigma} g_{\hat{\rho}\hat{\mu}} + \partial_{\hat{\lambda}}\partial_{\hat{\nu}}\partial_{\hat{\rho}} g_{\hat{\mu}\hat{\sigma}} \\ &\qquad + \partial_{\hat{\rho}}\partial_{\hat{\mu}}\partial_{\hat{\lambda}} g_{\hat{\sigma}\hat{\nu}} - \partial_{\hat{\rho}}\partial_{\hat{\mu}}\partial_{\hat{\sigma}} g_{\hat{\nu}\hat{\lambda}} - \partial_{\hat{\rho}}\partial_{\hat{\nu}}\partial_{\hat{\lambda}} g_{\hat{\sigma}\hat{\mu}} + \partial_{\hat{\rho}}\partial_{\hat{\nu}}\partial_{\hat{\sigma}} g_{\hat{\mu}\hat{\lambda}} \\ &\qquad + \partial_{\hat{\sigma}}\partial_{\hat{\mu}}\partial_{\hat{\rho}} g_{\hat{\lambda}\hat{\nu}} - \partial_{\hat{\sigma}}\partial_{\hat{\mu}}\partial_{\hat{\lambda}} g_{\hat{\nu}\hat{\rho}} - \partial_{\hat{\sigma}}\partial_{\hat{\nu}}\partial_{\hat{\rho}} g_{\hat{\lambda}\hat{\mu}} + \partial_{\hat{\sigma}}\partial_{\hat{\nu}}\partial_{\hat{\lambda}} g_{\hat{\mu}\hat{\rho}}) \\ &\quad = 0.\end{aligned} \tag{3.139}$$

Once again, since this is an equation between tensors it is true in any coordinate system, even though we derived it in a particular one. We recognize by now that the antisymmetry $R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}$ allows us to write this result as

$$\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0. \tag{3.140}$$

This is known as the **Bianchi identity**. For a general connection there would be additional terms involving the torsion tensor. It is closely related to the Jacobi

identity, since (recalling the definition of the Riemann tensor in terms of the commutator of covariant derivatives) it expresses

$$[[\nabla_\lambda, \nabla_\rho], \nabla_\sigma] + [[\nabla_\rho, \nabla_\sigma], \nabla_\lambda] + [[\nabla_\sigma, \nabla_\lambda], \nabla_\rho] = 0. \tag{3.141}$$

The Riemann tensor has four indices. At times it is useful to express a tensor as a sum of various pieces that are individually easier to handle and may have direct physical interpretations. The trick is to do this in a coordinate-invariant way. For example, we could decompose the Riemann tensor into $R^\rho{}_{\sigma ij}$ and $R^\rho{}_{\sigma i0}$, from which we could reconstruct the entire tensor (since $R^\rho{}_{\sigma 00}$ vanishes). But clearly this decomposition is not invariant under change of basis; we want to find a decomposition that is preserved when we change coordinates. What we are really doing is considering representations of the Lorentz group. We have two fundamental tricks at our disposal: taking contractions, and taking symmetric/antisymmetric parts. For example, given an arbitrary (0, 2) tensor $X_{\mu\nu}$, we can decompose it into its symmetric and antisymmetric pieces,

$$X_{\mu\nu} = X_{(\mu\nu)} + X_{[\mu\nu]}, \tag{3.142}$$

and the symmetric part can be further decomposed into its trace $X = g^{\mu\nu}X_{(\mu\nu)}$ and a trace-free part $\widehat{X}_{\mu\nu} = X_{(\mu\nu)} - \frac{1}{n}Xg_{\mu\nu}$, so that

$$X_{\mu\nu} = \frac{1}{n}Xg_{\mu\nu} + \widehat{X}_{\mu\nu} + X_{[\mu\nu]}. \tag{3.143}$$

(Note that $X_{[\mu\nu]}$ is automatically traceless.) When we change coordinates, the different pieces $Xg_{\mu\nu}$, $\widehat{X}_{\mu\nu}$, and $X_{[\mu\nu]}$ are rotated into themselves, not into each other; we say that they define "invariant subspaces" of the space of (0, 2) tensors. For more complicated tensors the equivalent decomposition might not be so simple.

For the Riemann tensor, our first step is to take a contraction to form the **Ricci tensor**:

$$\boxed{R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}.} \tag{3.144}$$

For the curvature tensor formed from an arbitrary (not necessarily Christoffel) connection, there are a number of independent contractions to take. Our primary concern is with the Christoffel connection, for which (3.144) is the only independent contraction; all others either vanish, or are related to this one. The Ricci tensor associated with the Christoffel connection is automatically symmetric,

$$\boxed{R_{\mu\nu} = R_{\nu\mu},} \tag{3.145}$$

as a consequence of the symmetries of the Riemann tensor. The trace of the Ricci tensor is the **Ricci scalar** (or **curvature scalar**):

$$\boxed{R = R^{\mu}{}_{\mu} = g^{\mu\nu} R_{\mu\nu}.} \tag{3.146}$$

We could also form the trace-free part $\widehat{R}_{\mu\nu} = R_{\mu\nu} - \frac{1}{n} R g_{\mu\nu}$, but this turns out not to be especially useful; it is more common to express things in terms of $R_{\mu\nu}$ and R.

The Ricci tensor and scalar contain all of the information about traces of the Riemann tensor, leaving us the trace-free parts. These are captured by the **Weyl tensor**, which is basically the Riemann tensor with all of its contractions removed. It is given in n dimensions by

$$\begin{aligned} C_{\rho\sigma\mu\nu} = R_{\rho\sigma\mu\nu} &- \frac{2}{(n-2)} \left(g_{\rho[\mu} R_{\nu]\sigma} - g_{\sigma[\mu} R_{\nu]\rho} \right) \\ &+ \frac{2}{(n-1)(n-2)} g_{\rho[\mu} g_{\nu]\sigma} R \end{aligned} \tag{3.147}$$

This messy formula is designed so that all possible contractions of $C_{\rho\sigma\mu\nu}$ vanish, while it retains the symmetries of the Riemann tensor:

$$\begin{aligned} C_{\rho\sigma\mu\nu} &= C_{[\rho\sigma][\mu\nu]}, \\ C_{\rho\sigma\mu\nu} &= C_{\mu\nu\rho\sigma}, \\ C_{\rho[\sigma\mu\nu]} &= 0. \end{aligned} \tag{3.148}$$

The Weyl tensor is only defined in three or more dimensions, and in three dimensions it vanishes identically. One of the most important properties of the Weyl tensor is that it is invariant under conformal transformations (discussed in Appendix G). This means that if you compute $C^{\rho}{}_{\sigma\mu\nu}$ (note that the first index is upstairs) for some metric $g_{\mu\nu}$, and then compute it again for a metric given by $\omega^2(x) g_{\mu\nu}$, where $\omega(x)$ is an arbitrary nonvanishing function of spacetime, you get the same answer. For this reason it is often known as the *conformal tensor*.

An especially useful form of the Bianchi identity comes from contracting twice on (3.139):

$$\begin{aligned} 0 &= g^{\nu\sigma} g^{\mu\lambda} (\nabla_{\lambda} R_{\rho\sigma\mu\nu} + \nabla_{\rho} R_{\sigma\lambda\mu\nu} + \nabla_{\sigma} R_{\lambda\rho\mu\nu}) \\ &= \nabla^{\mu} R_{\rho\mu} - \nabla_{\rho} R + \nabla^{\nu} R_{\rho\nu}, \end{aligned} \tag{3.149}$$

or

$$\nabla^{\mu} R_{\rho\mu} = \tfrac{1}{2} \nabla_{\rho} R. \tag{3.150}$$

Notice that, unlike the partial derivative, it makes sense to raise an index on the covariant derivative, due to metric compatibility. We define the **Einstein tensor**

as

$$\boxed{G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu}.} \tag{3.151}$$

In four dimensions the Einstein tensor can be thought of as a trace-reversed version of the Ricci tensor. We then see that the twice-contracted Bianchi identity (3.150) is equivalent to

$$\boxed{\nabla^{\mu}G_{\mu\nu} = 0.} \tag{3.152}$$

The Einstein tensor, which is symmetric due to the symmetry of the Ricci tensor and the metric, will be of great importance in general relativity.

We should pause at this point to contrast the formalism we have developed with our intuitive notion of curvature. Our intuition is unfortunately contaminated by the fact that we are used to thinking about one- and two-dimensional spaces embedded in the (almost) Euclidean space in which we live. We think, for example, of a straight line as having no curvature, while a circle (S^1) is curved. However, according to (3.137), in one, two, three, and four dimensions there are 0, 1, 6 and 20 independent components of the Riemann tensor, respectively. (Everything we say about the curvature in these examples refers to the curvature associated with the Christoffel connection, and therefore the metric.) Therefore it is impossible for a one-dimensional space such as S^1 to have any curvature as we have defined it. The apparent contradiction stems from the fact that our intuitive notion of curvature depends on the extrinsic geometry of the manifold, which characterizes how a space is embedded in some larger space, while the Riemann curvature is a property of the intrinsic geometry of a space, which could be measured by observers confined to the manifold. Beings that lived on a circle and had no access to the larger world would necessarily think that they lived in a flat geometry—for example, there is no possibility of a nondegenerate infinitesimal loop around which we could parallel-transport a vector and have it come back rotated from its original position. Extrinsic curvature, discussed in Appendix D, is occasionally useful in GR when we wish to describe submanifolds of spacetime; but most often we are interested in the intrinsic geometry of spacetime itself, which does not rely on any embeddings.

We can illustrate the intrinsic/extrinsic difference further with an example from two dimensions, where the curvature has one independent component. In fact, all of the information about the curvature is contained in the single component of the Ricci scalar. Consider a torus, portrayed in Figure 3.7, which can be thought of as a square region of the plane with opposite sides identified (topologically, $S^1 \times S^1$). Although a torus embedded in three dimensions looks curved from our point of view, it should be clear that we can put a metric on the torus whose components are constant in an appropriate coordinate system—simply unroll it and use the Euclidean metric of the plane, $ds^2 = \mathrm{d}x^2 + \mathrm{d}y^2$. In this metric, the torus is flat.

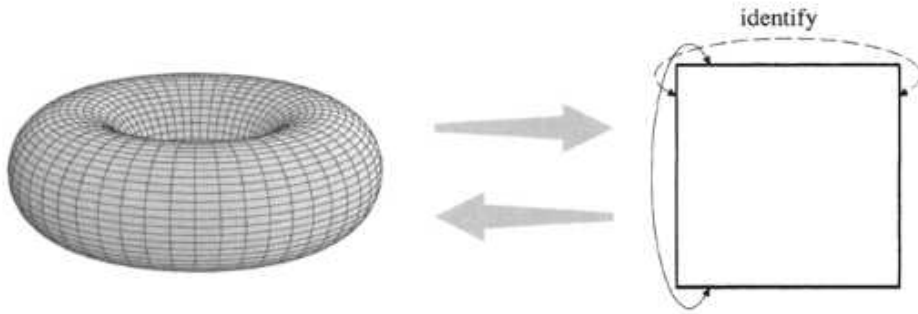


FIGURE 3.7 A torus thought of as a square in flat space with opposite sides identified.

There is also nothing to stop us from introducing a different metric in which the torus is not flat, but the point we are trying to emphasize is that it can be made flat in some metric. Every time we embed a manifold in a larger space, the manifold inherits an "induced metric" from the background in which it is embedded, as discussed in the Appendix A. Our point here is that a torus embedded in a flat three-dimensional Euclidean space will have an induced metric that is curved, but we can nevertheless choose to put a different metric on it so that the intrinsic geometry is flat.

Let's turn to a simple example where the curvature does not vanish. We have already talked about the two-sphere S^2, with metric

$$ds^2 = a^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{3.153}$$

where a is the radius of the sphere. It will actually be the radius if our sphere is embedded in $\mathbf{R}^3$, but we can call it the radius even in the absence of any embedding. Two-dimensional people living on the sphere could calculate a by measuring the area of the sphere, dividing by 4π, and taking the square root; using the word "radius" to refer to this quantity is merely a convenience. We should also point out that the notion of a sphere is sometimes used in the weaker topological sense, without any particular metric being assumed; the metric we are using is called the *round metric*. Without going through the details, the nonzero connection coefficients for (3.153) are

$$\begin{aligned} \Gamma^{\theta}_{\phi\phi} &= -\sin\theta\cos\theta \\ \Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} &= \cot\theta. \end{aligned} \tag{3.154}$$

Let's compute a promising component of the Riemann tensor:

$$\begin{aligned} R^{\theta}{}_{\phi\theta\phi} &= \partial_\theta\Gamma^{\theta}_{\phi\phi} - \partial_\phi\Gamma^{\theta}_{\theta\phi} + \Gamma^{\theta}_{\theta\lambda}\Gamma^{\lambda}_{\phi\phi} - \Gamma^{\theta}_{\phi\lambda}\Gamma^{\lambda}_{\theta\phi} \\ &= (\sin^2\theta - \cos^2\theta) - (0) + (0) - (-\sin\theta\cos\theta)(\cot\theta) \\ &= \sin^2\theta. \end{aligned} \tag{3.155}$$

The notation is obviously imperfect, since the Greek letter λ is a dummy index that is summed over, while the Greek letters θ and ϕ represent specific coordinates. Lowering an index, we have

$$\begin{aligned} R_{\theta\phi\theta\phi} &= g_{\theta\lambda} R^{\lambda}{}_{\phi\theta\phi} \\ &= g_{\theta\theta} R^{\theta}{}_{\phi\theta\phi} \\ &= a^2 \sin^2\theta. \end{aligned} \tag{3.156}$$

It is easy to check that all of the components of the Riemann tensor either vanish or are related to this one by symmetry. We can go on to compute the Ricci tensor via $R_{\mu\nu} = g^{\alpha\beta} R_{\alpha\mu\beta\nu}$. We obtain

$$\begin{aligned} R_{\theta\theta} &= g^{\phi\phi} R_{\phi\theta\phi\theta} = 1 \\ R_{\theta\phi} &= R_{\phi\theta} = 0 \\ R_{\phi\phi} &= g^{\theta\theta} R_{\theta\phi\theta\phi} = \sin^2\theta. \end{aligned} \tag{3.157}$$

The Ricci scalar is similarly straightforward:

$$R = g^{\theta\theta} R_{\theta\theta} + g^{\phi\phi} R_{\phi\phi} = \frac{2}{a^2}. \tag{3.158}$$

Therefore the Ricci scalar, which for a two-dimensional manifold completely characterizes the curvature, is a constant over the two-sphere. If we had perturbed the metric (corresponding physically to bumps on the sphere), this would no longer have been the case. Note that the scalar curvature decreases as the radius of the sphere increases. Even in more general contexts, we will sometimes refer to the "radius of curvature" of a manifold as providing a length scale over which the curvature varies; the larger the radius of curvature, the smaller the curvature itself.

3.8 ■ SYMMETRIES AND KILLING VECTORS

The real world is a messy place, and we have no hope of finding a metric that describes our actual universe, or even any small part thereof, with perfect precision. Instead, we model spacetime via various approximations appropriate to the physical situation being studied. For example, the geometry outside a star or planet may be approximated, to some order of precision, as being spherically symmetric, even if the real situation includes small deviations from this symmetry—these may be added in later as perturbations.

General relativity is no different from other fields of physics, then, in being especially interested in solutions with symmetry. In fact, such properties may be even more crucial in GR than in, say, electromagnetism, since the nonlinear nature of Einstein's equation (discussed in the next chapter) makes it hard to find any exact solutions at all. In the context of curved spacetime, however, we need to be

more careful than usual about what exactly is meant by "symmetry." In this section we develop some useful tools for studying symmetry; a deeper investigation can be found in Appendix B.

We think of a manifold M as possessing a symmetry if the geometry is invariant under a certain transformation that maps M to itself; that is, if the metric is the same, in some sense, from one point to another. In fact different tensor fields may possess different symmetries; symmetries of the metric are called **isometries**. Sometimes the existence of isometries is obvious; consider, for example, four-dimensional Minkowski space,

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -dt^2 + dx^2 + dy^2 + dz^2. \tag{3.159}$$

We know of several isometries of this space; these include translations ($x^\mu \to x^\mu + a^\mu$, with a^μ fixed) and Lorentz transformations ($x^\mu \to \Lambda^\mu{}_\nu x^\nu$, with $\Lambda^\mu{}_\nu$ a Lorentz-transformation matrix). The fact that the metric is invariant under translations is made immediately apparent by the simple fact that the metric coefficients $\eta_{\mu\nu}$ are independent of the individual coordinate functions x^μ. Indeed, whenever $\partial_{\sigma_*} g_{\mu\nu} = 0$ for some fixed σ_* (but for all μ and ν), there will be a symmetry under translations along x^{σ_*}:

$$\partial_{\sigma_*} g_{\mu\nu} = 0 \quad \Rightarrow \quad x^{\sigma_*} \to x^{\sigma_*} + a^{\sigma_*} \text{ is a symmetry.} \tag{3.160}$$

The careful reader will have noticed that we still haven't precisely defined what we mean by symmetry; roughly we imagine that the metric is invariant under some transformation, but the precise meaning is only developed in Appendix B. Also, the implication arrow in (3.160) only goes one way, and it would be nice to have a clean criterion for deciding when a given transformation counts as a symmetry; this will come soon.

Isometries of the form (3.160) have immediate consequences for the motion of test particles as described by the geodesic equation. Recall from (3.61) that the geodesic equation can be written in terms of the four-momentum $p^\mu = mU^\mu$ (valid for timelike paths, at least) as

$$p^\lambda \nabla_\lambda p^\mu = 0. \tag{3.161}$$

By metric compatibility we are free to lower the index μ, and then we may expand the covariant derivative to obtain

$$p^\lambda \partial_\lambda p_\mu - \Gamma^\sigma_{\lambda\mu} p^\lambda p_\sigma = 0. \tag{3.162}$$

The first term tells us how the momentum components change along the path,

$$p^\lambda \partial_\lambda p_\mu = m \frac{dx^\lambda}{d\tau} \partial_\lambda p_\mu = m \frac{dp_\mu}{d\tau}, \tag{3.163}$$

while the second term is

$$\Gamma^{\sigma}_{\lambda\mu}p^{\lambda}p_{\sigma} = \tfrac{1}{2}g^{\sigma\nu}(\partial_{\lambda}g_{\mu\nu} + \partial_{\mu}g_{\nu\lambda} - \partial_{\nu}g_{\lambda\mu})p^{\lambda}p_{\sigma} \tag{3.164}$$

$$= \tfrac{1}{2}(\partial_{\lambda}g_{\mu\nu} + \partial_{\mu}g_{\nu\lambda} - \partial_{\nu}g_{\lambda\mu})p^{\lambda}p^{\nu} \tag{3.165}$$

$$= \tfrac{1}{2}(\partial_{\mu}g_{\nu\lambda})p^{\lambda}p^{\nu}, \tag{3.166}$$

where we have used the symmetry of $p^{\lambda}p^{\nu}$ to go from the second line to the third. So, without yet making any assumptions about symmetry, we see that the geodesic equation can be written as

$$m\frac{dp_{\mu}}{d\tau} = \frac{1}{2}(\partial_{\mu}g_{\nu\lambda})p^{\lambda}p^{\nu}. \tag{3.167}$$

Therefore, if all of the metric coefficients are independent of the coordinate x^{σ_*}, we find that this isometry implies that the momentum component p_{σ_*} is a conserved quantity of the motion:

$$\partial_{\sigma_*}g_{\mu\nu} = 0 \quad \Rightarrow \quad \frac{dp_{\sigma_*}}{d\tau} = 0. \tag{3.168}$$

This will hold along any geodesic, even though we only derived it for timelike ones. The conserved quantities implied by isometries are extremely useful in studying the motion of test particles in curved backgrounds.

Of course, even though independence of the metric components on one or more coordinates implies the existence of isometries, the converse does not necessarily hold. Symmetry under Lorentz transformations, for example, is not manifest as independence of $\eta_{\mu\nu}$ on any coordinates; indeed, in four dimensions, there are four types of translations and six types of Lorentz transformations, for a total of ten, which is obviously larger than the number of dimensions the metric could possibly be independent of. What is more, it would be simple enough to transform to a complicated coordinate system where not even the translational symmetries were obvious. Such a coordinate transformation would change the metric components, but not the underlying geometry, which is what the symmetry is really characterizing. Clearly a more systematic procedure is called for.

We can develop such a procedure by casting the right-hand equation of (3.168), expressing constancy of one of the components of the momentum, in a more manifestly covariant form. If x^{σ_*} is the coordinate which $g_{\mu\nu}$ is independent of, let us consider the vector ∂_{σ_*}, which we label as K:

$$K = \partial_{\sigma_*}, \tag{3.169}$$

which is equivalent in component notation to

$$K^{\mu} = (\partial_{\sigma_*})^{\mu} = \delta^{\mu}_{\sigma_*}. \tag{3.170}$$

We say that the vector K^{μ} generates the isometry; this means that the transformation under which the geometry is invariant is expressed infinitesimally as a motion

in the direction of K^μ. Again, the notion is developed more fully in Appendix B. In terms of this vector, the noncovariant-looking quantity p_{σ_*} is simply

$$p_{\sigma_*} = K^\nu p_\nu = K_\nu p^\nu. \tag{3.171}$$

Meanwhile, the constancy of this (scalar) quantity along the path is equivalent to the statement that its directional derivative along the geodesic vanishes:

$$\frac{dp_{\sigma_*}}{d\tau} = 0 \quad \leftrightarrow \quad p^\mu \nabla_\mu (K_\nu p^\nu) = 0. \tag{3.172}$$

Expanding the expression on the right, we obtain

$$\begin{aligned} p^\mu \nabla_\mu (K_\nu p^\nu) &= p^\mu K_\nu \nabla_\mu p^\nu + p^\mu p^\nu \nabla_\mu K_\nu \\ &= p^\mu p^\nu \nabla_\mu K_\nu \\ &= p^\mu p^\nu \nabla_{(\mu} K_{\nu)}, \end{aligned} \tag{3.173}$$

where in the second line we have invoked the geodesic equation ($p^\mu \nabla_\mu p^\nu = 0$). In the third line we have used the fact that $p^\mu p^\nu$ is automatically symmetric in μ and ν, so only the symmetric part of $\nabla_\mu K_\nu$ could possibly contribute. We therefore conclude that any vector K_μ that satisfies $\nabla_{(\mu} K_{\nu)} = 0$ implies that $K_\nu p^\nu$ is conserved along a geodesic trajectory:

$$\boxed{\nabla_{(\mu} K_{\nu)} = 0 \quad \Rightarrow \quad p^\mu \nabla_\mu (K_\nu p^\nu) = 0.} \tag{3.174}$$

The equation on the left is known as **Killing's equation**, and vector fields that satisfy it are known as **Killing vector fields** (or simply Killing vectors). You can verify for yourself that, if the metric is independent of some coordinate x^{σ_*}, the vector ∂_{σ_*} will satisfy Killing's equation. In fact, if a vector K^μ satisfies Killing's equation, it will always be possible to find a coordinate system in which $K = \partial_{\sigma_*}$; but in general we cannot find coordinates in which all the Killing vectors are simultaneously of this form, nor is this form necessary for the vector to satisfy Killing's equation.

As we investigate in Appendix B, Killing vector fields on a manifold are in one-to-one correspondence with continuous symmetries of the metric on that manifold. Every Killing vector implies the existence of conserved quantities associated with geodesic motion. This can be understood physically: by definition the metric is unchanging along the direction of the Killing vector. Loosely speaking, therefore, a free particle will not feel any forces in this direction, and the component of its momentum in that direction will consequently be conserved. In fact, the same kind of logic by which we showed that $K_\nu p^\nu$ is conserved along a geodesic if $\nabla_{(\mu} K_{\nu)} = 0$ generalizes to additional indices: a **Killing tensor** is a symmetric l-index tensor $K_{\nu_1 \cdots \nu_l}$ that satisfies the obvious generalization of Killing's equation, and correspondingly leads to conserved quantities by contracting with l copies of

the momentum:

$$\nabla_{(\mu} K_{\nu_1 \cdots \nu_l)} = 0 \quad \Rightarrow \quad p^\mu \nabla_\mu (K_{\nu_1 \cdots \nu_l} p^{\nu_1} \cdots p^{\nu_l}) = 0. \tag{3.175}$$

Simple examples of Killing tensors are the metric itself, and symmetrized tensor products of Killing vectors. Killing tensors are not related in a simple way to symmetries of the spacetime, but they will simplify our analysis of rotating black holes and expanding universes.

Derivatives of Killing vectors can be related to the Riemann tensor by

$$\nabla_\mu \nabla_\sigma K^\rho = R^\rho{}_{\sigma\mu\nu} K^\nu, \tag{3.176}$$

as you are asked to prove in the exercises. Contracting this expression yields

$$\nabla_\mu \nabla_\sigma K^\mu = R_{\sigma\nu} K^\nu. \tag{3.177}$$

These relations, along with the Bianchi identity and Killing's equation, suffice to show that the directional derivative of the Ricci scalar along a Killing vector field will vanish,

$$K^\lambda \nabla_\lambda R = 0. \tag{3.178}$$

This last fact is another reflection of the idea that the geometry is not changing along a Killing vector field.

Besides leading to conserved quantities for the motion of individual particles, the existence of a timelike Killing vector allows us to define a conserved energy for the entire spacetime. Given a Killing vector K_ν and a conserved energy-momentum tensor $T_{\mu\nu}$, we can construct a current

$$J^\mu_T = K_\nu T^{\mu\nu} \tag{3.179}$$

that is automatically conserved,

$$\begin{aligned} \nabla_\mu J^\mu_T &= (\nabla_\mu K_\nu) T^{\mu\nu} + K_\nu (\nabla_\mu T^{\mu\nu}) \\ &= 0. \end{aligned} \tag{3.180}$$

The first term vanishes by virtue of Killing's equation (since the symmetry of the upper indices serves to automatically symmetrize the lower indices), and the second term vanishes by conservation of $T_{\mu\nu}$. If K_ν is timelike, we can integrate over a spacelike hypersurface Σ to define the total energy,

$$E_T = \int_\Sigma J^\mu_T n_\mu \sqrt{\gamma} d^3x, \tag{3.181}$$

where γ_{ij} is the induced metric on Σ and n_μ is the normal vector to Σ. In Appendix E we discuss integration over hypersurfaces, and in particular Stokes's theorem; as explained there, E_T will be the same when integrated over any spacelike

hypersurface, and is therefore conserved. This result fits nicely with our discussion in Section 3.5, where we found that the total energy is not typically conserved in an expanding universe; expansion means that the metric is changing with time, so there is no isometry in this direction. When there is a timelike Killing vector, we can write the metric in a form where it is independent of the timelike coordinate, and Noether's theorem implies a conserved energy. Similarly, spacelike Killing vectors may be used to construct conserved momenta (or angular momenta).

Although it may or may not be simple to actually solve Killing's equation in any given spacetime, it is frequently possible to write down some Killing vectors by inspection. (Of course a generic metric has no Killing vectors at all, but to keep things simple we often deal with metrics with high degrees of symmetry.) For example, in $\mathbf{R}^3$ with metric $ds^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$, independence of the metric components with respect to x, y, and z immediately yields three Killing vectors:

$$\begin{aligned} X^\mu &= (1, 0, 0) \\ Y^\mu &= (0, 1, 0) \\ Z^\mu &= (0, 0, 1). \end{aligned} \tag{3.182}$$

These clearly represent the three translations. There are also three rotational symmetries in $\mathbf{R}^3$, which are not quite as simple. To find them, imagine first going to polar coordinates,

$$\begin{aligned} x &= r \sin\theta \cos\phi \\ y &= r \sin\theta \sin\phi \\ z &= r \cos\theta, \end{aligned} \tag{3.183}$$

where the metric takes the form

$$ds^2 = \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 + r^2 \sin^2\theta \, \mathrm{d}\phi^2. \tag{3.184}$$

Now the metric (the *same* metric, just in a different coordinate system) is manifestly independent of ϕ. We therefore know that $R = \partial_\phi$ is a Killing vector. Transforming back to Cartesian coordinates, this becomes

$$R = -y\partial_x + x\partial_y. \tag{3.185}$$

The Cartesian components R^μ are therefore $(-y, x, 0)$. Since this represents a rotation about the z-axis, it is straightforward to guess the components of all three rotational Killing vectors:

$$\begin{aligned} R^\mu &= (-y,\ x,\ 0) \\ S^\mu &= (\ z,\ 0, -x) \\ T^\mu &= (\ 0, -z,\ y). \end{aligned} \tag{3.186}$$

representing rotations about the z, y, and x-axes, respectively. You can check for yourself that these actually do solve Killing's equation. The overall signs don't matter, since minus a Killing vector is still a Killing vector.

This exercise leads directly to the Killing vectors for the two-sphere S^2 with metric

$$ds^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2. \tag{3.187}$$

Since this sphere can be thought of as the locus of points at unit distance from the origin in $\mathbf{R}^3$, and the rotational Killing vectors all rotate such a sphere into itself, they also represent symmetries of S^2. To get explicit coordinate-basis representations for these vectors, we first transform the three-dimensional vectors (3.186) to polar coordinates $x^{\mu'} = (r, \theta, \phi)$. A straightforward calculation reveals

$$\begin{aligned} R &= \partial_\phi \\ S &= \cos\phi\,\partial_\theta - \cot\theta\sin\phi\,\partial_\phi \\ T &= -\sin\phi\,\partial_\theta - \cot\theta\cos\phi\,\partial_\phi. \end{aligned} \tag{3.188}$$

Notice that there are no components along ∂_r, which makes sense for a rotational isometry. Therefore the expressions (3.188) for the three rotational Killing vectors in $\mathbf{R}^3$ are exactly the same as those of S^2 in spherical polar coordinates.

In $n \geq 2$ dimensions, there can be more Killing vectors than dimensions. This is because a set of Killing vector fields can be linearly independent, even though at any one point on the manifold the vectors at that point are linearly dependent. It is trivial to show (so you should do it yourself) that a linear combination of Killing vectors with *constant* coefficients is still a Killing vector (in which case the linear combination does not count as an independent Killing vector), but this is not generally true with coefficients that vary over the manifold. You can also show that the commutator of two Killing vector fields is a Killing vector field; this is very useful to know, but it may be the case that the commutator gives you a vector field that is not linearly independent (or it may simply vanish). The problem of finding all of the Killing vectors of a metric is therefore somewhat tricky, as it is not always clear when to stop looking.

3.9 ■ MAXIMALLY SYMMETRIC SPACES

How symmetric can a space possibly be? An example of a space with the highest possible degree of symmetry is $\mathbf{R}^n$ with the flat Euclidean metric. Consider the isometries of this space, which we know to be translations and rotations in n dimensions, from the perspective of what they do in the neighborhood of some fixed point p. The translations are those transformations that move the point; there are n independent axes along which it can be moved, and hence n total translations. The rotations, centered at p, are those transformations that leave p invariant; they

can be thought of as moving one of the axes through p into one of the others. There are n axes, and for each axis there are $n-1$ other axes into which it can be rotated, but we shouldn't count a rotation of y into x as separate from a rotation of x into y, so the total number of independent rotations is $\frac{1}{2}n(n-1)$. We therefore have

$$n + \tfrac{1}{2}n(n-1) = \tfrac{1}{2}n(n+1) \tag{3.189}$$

independent symmetries of $\mathbf{R}^n$. But our counting argument only referred to the behavior of the symmetry in a neighborhood of p, not globally all over the manifold; so even in the presence of curvature the counting should be the same. If the metric signature is not Euclidean, some of the rotations will actually be boosts, but again the counting will be the same. The number of isometries is, of course, the number of linearly independent Killing vector fields. We therefore refer to an n-dimensional manifold with $\frac{1}{2}n(n+1)$ Killing vectors as a **maximally symmetric space**. The most familiar examples of maximally symmetric spaces are n-dimensional Euclidean spaces $\mathbf{R}^n$ and the n-dimensional spheres S^n. For an n-dimensional sphere we usually think of the isometries as consisting of $\frac{1}{2}n(n+1)$ independent rotations, rather than as some collection of both rotations and translations. However, if we consider the action of these rotations on some fixed point p, a moment's thought convinces us that the entire set can be decomposed into $\frac{1}{2}n(n-1)$ rotations around the point (keeping p fixed), and another n that move p along each direction, just as in $\mathbf{R}^n$.

If a manifold is maximally symmetric, the curvature is the same everywhere (as expressed by translation-like isometries) and the same in every direction (as expressed by rotation-like isometries). Hence, if we know the curvature of a maximally symmetric space at one point, we know it everywhere. Indeed, there are only a small number of possible maximally symmetric spaces; they are classified by the curvature scalar R (which will be constant everywhere), the dimensionality n, the metric signature, and perhaps some discrete pieces of information relating to the global topology (distinguishing, for example, an n-torus from $\mathbf{R}^n$, and tori of different sizes from each other). It follows that we should be able to reconstruct the entire Riemann tensor of such a space from the Ricci scalar R; let's see how this works.

The basic idea is simply that, since the geometry looks the same in all directions, the curvature tensor should look the same in all directions. What might this mean? First choose locally inertial coordinates at some point p, so that $g_{\hat{\mu}\hat{\nu}} = \eta_{\hat{\mu}\hat{\nu}}$. Of course, locally inertial coordinates are not unique; for example, we can perform a Lorentz transformation at p and the metric components will remain those of $\eta_{\hat{\mu}\hat{\nu}}$. (By "doing a Lorentz transformation" we really are referring to a change of basis vectors in T_p; in a curved spacetime, this only makes sense at a single point, not over a region.) Since the geometry is maximally symmetric, we want the same to be true of the Riemann tensor; that is, the components of $R_{\hat{\rho}\hat{\sigma}\hat{\mu}\hat{\nu}}$ should not change under a Lorentz transformation either, since there is no preferred direction in spacetime. But there are unique tensors that do not change their components under Lorentz transformations—the metric, the Kronecker delta, and

the Levi–Civita tensor. This means that, in these coordinates and at this point, the components of $R_{\hat{\rho}\hat{\sigma}\hat{\mu}\hat{\nu}}$ will be proportional to a tensor constructed from these invariant tensors. Attempting to match the symmetries of the Riemann tensor reveals that there is a unique possibility:

$$R_{\hat{\rho}\hat{\sigma}\hat{\mu}\hat{\nu}} \propto g_{\hat{\rho}\hat{\mu}}g_{\hat{\sigma}\hat{\nu}} - g_{\hat{\rho}\hat{\nu}}g_{\hat{\sigma}\hat{\mu}}. \tag{3.190}$$

But this is a completely tensorial relation, so it must be true in any coordinate system. We have argued in favor of this relation at a single point p, but in a maximally symmetric space all points are created equal, so it must also be true at any other point as well. The proportionality constant is easily fixed by contracting both sides twice [the left-hand side becomes R, and the right-hand side is $n(n-1)$]. We end up with an equation true in any maximally symmetric space, at any point, in any coordinate system:

$$R_{\rho\sigma\mu\nu} = \frac{R}{n(n-1)}(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}). \tag{3.191}$$

Likewise, if the Riemann tensor satisfies this condition (with R a constant over the manifold), the metric will be maximally symmetric. In two dimensions, finding that R is a constant is sufficient to prove that a space is maximally symmetric, since there is only one independent component of the curvature. In higher dimensions you have to work harder.

Locally, then (ignoring questions of global topology), a maximally symmetric space of given dimension and signature is fully specified by R. The basic classification of such spaces is simply whether R is positive, zero, or negative, since the magnitude of R represents an overall scaling of the size of the space. For Euclidean signatures, the flat maximally symmetric spaces are planes or appropriate higher-dimensional generalizations, while the positively curved ones are spheres. Maximally symmetric Euclidean spaces of negative curvature are hyperboloids, denoted H^n. These are less familiar because even a two-dimensional hyperboloid cannot be isometrically embedded in $\mathbf{R}^3$. Let's examine this two-dimensional hyperboloid briefly.

There are a number of ways of representing H^2, which has the same topology as $\mathbf{R}^2$. One simple way is as the **Poincaré half-plane**, which is the region $y > 0$ of a two-dimensional region with coordinates $\{x, y\}$ and metric

$$ds^2 = \frac{a^2}{y^2}(\mathrm{d}x^2 + \mathrm{d}y^2). \tag{3.192}$$

The geometry of the Poincaré half-plane is of course different from that of the upper half of $\mathbf{R}^2$, despite the use of similar coordinates. For example, we can compute the length of a line segment stretching vertically (x = constant) from y_1 to y_2:

$$\begin{aligned}\Delta s &= \int_{y_1}^{y_2} \sqrt{g_{\mu\nu}\frac{dx^\mu}{dy}\frac{dx^\nu}{dy}}\,dy \\ &= a\int_{y_1}^{y_2}\frac{dy}{y} \\ &= a\ln\left(\frac{y_2}{y_1}\right).\end{aligned} \tag{3.193}$$

This is not at all the result $\Delta s = y_2 - y_1$ we would expect in Euclidean space. In particular, notice that the path length becomes infinite for paths that approach the boundary $y = 0$. In other words, it's not really a boundary at all; it's infinitely far away, as far as anyone living on the hyperboloid is concerned.

The nonvanishing Christoffel symbols for (3.192) are

$$\begin{aligned}\Gamma^x_{xy} = \Gamma^x_{yx} &= -y^{-1} \\ \Gamma^y_{xx} &= y^{-1} \\ \Gamma^y_{yy} &= -y^{-1}.\end{aligned} \tag{3.194}$$

From these it is straightforward to show that geodesics satisfy

$$(x - x_0)^2 + y^2 = l^2, \tag{3.195}$$

for some constants x_0 and l. Curves of this form are semicircles with centers located on the x-axis, as shown in Figure 3.8. In the limit as $x_0 \to \infty$ and $l \to \infty$ with $l - x_0$ fixed, we get a straight vertical line. Following our discussion of S^2 at the end of Section 3.7, we calculate a representative component of the Riemann tensor to be

$$R^x{}_{yxy} = -y^{-2}. \tag{3.196}$$

As with the two-sphere, all other components are either vanishing or related to this by symmetries. This is simply a reflection of the fact that we are in two di-

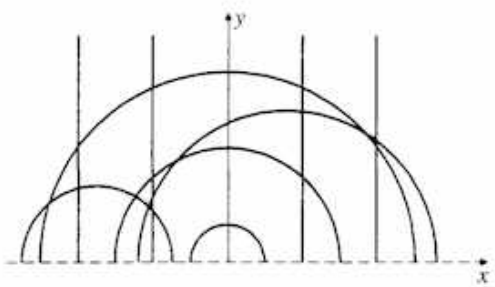


FIGURE 3.8 The upper half plane with a negatively curved metric. Geodesics are semicircles and straight lines that intersect the x-axis vertically.

mensions, with only one independent component of curvature. Turning the crank yields the Ricci tensor,

$$\begin{aligned} R_{xx} &= -y^{-2} \\ R_{xy} &= 0 \\ R_{yy} &= -y^{-2}, \end{aligned} \tag{3.197}$$

and the curvature scalar,

$$R = -\frac{2}{a^2}. \tag{3.198}$$

We see that it matches that of S^2 with the opposite sign, and in particular that it is a constant. Since we are in two dimensions, this is enough to ensure that our metric really is maximally symmetric. Of course there are coordinates in which H^2 looks very different; one is introduced in the Exercises.

Locally, then, a maximally symmetric space of Euclidean signature is either a plane, a sphere, or a hyperboloid, depending on the sign of R. Globally, any maximally symmetric space (of Euclidean signature) can be constructed by taking a carefully chosen region of one of these three spaces and identifying different sides, as the flat torus can be constructed from $\mathbf{R}^2$. As an aside, let's briefly mention a connection between local geometry and global topology, encompassed by the Gauss–Bonnet theorem. For a two-dimensional compact boundaryless orientable manifold, this reads

$$\chi(M) = \frac{1}{4\pi} \int_M R\sqrt{|g|}\, d^n x, \tag{3.199}$$

where $\chi(M)$ is a topological invariant of the space, known as the Euler characteristic. In general it can be calculated from the cohomology spaces mentioned in Chapter 2; in two dimensions, however, it is simply given by

$$\chi(M) = 2(1 - g), \tag{3.200}$$

where g is the genus of the surface (zero for a sphere, and equal to the number of handles of a torus or Riemann surface). The Gauss–Bonnet theorem holds whether or not the curvature R is a constant; when it is, however, we see that all Riemann surfaces of genus $g \geq 2$ must have negative curvature, just as a sphere must be positively curved and a torus must be flat.

Continuing our aside, think for the moment about string theory, which claims that the fundamental objects comprising the universe are small one-dimensional loops of string. Such strings have two-dimensional "world-sheets" rather than one-dimensional worldlines. Doing perturbation theory in string theory (the equivalent of calculating Feynman diagrams in quantum field theory) involves summing over all world-sheet geometries (generally, for technical reasons, Eu-

clidean geometries). This sounds like a lot of geometries, but in two dimensions any metric can be written as some fiducial metric times a conformal factor. This should be plausible, since there is only one curvature component; you are asked to prove it in the Exercises. The fiducial metric can be chosen differently for each world-sheet topology, and we can make our lives easier by choosing it to be (locally) a metric of maximal symmetry—the round sphere for genus zero, the plane for genus one, and the hyperboloid for higher genera. Even more fortunately, the string theories of greatest physical interest are the so-called critical string theories, for which the conformal factor itself doesn't matter. This is one of the things that makes doing calculations in perturbative string theory possible; we only have to sum over a discrete set of topologies, with a finite number of modular parameters for each topology (such as the parameters telling us the sizes of the different directions in a torus).

We close this section with one last point. We have explored the maximally symmetric spaces of Euclidean signatures; there are, of course, corresponding spacetimes with Lorentzian signatures. We know that the maximally symmetric spacetime with $R = 0$ is simply Minkowski space. The positively curved maximally symmetric spacetime is called de Sitter space, while that with negative curvature is imaginatively labeled anti-de Sitter space. These spacetimes will be more thoroughly discussed in Chapter 8.

It should be clear by now that the Appendices flesh out these ideas in important ways. Impatient readers may skip over them, but it would be a shame to do so.

3.10 ■ GEODESIC DEVIATION

The Riemann tensor shows up as a consequence of curvature in one more way: geodesic deviation. You have undoubtedly heard that the defining property of Euclidean (flat) geometry is the parallel postulate: initially parallel lines remain parallel forever. Of course in a curved space this is not true; on a sphere, certainly, initially parallel geodesics will eventually cross. We would like to quantify this behavior for an arbitrary curved space.

The problem is that the notion of "parallel" does not extend naturally from flat to curved spaces. The best we can do is to consider geodesic curves that might be initially parallel, and see how they behave as we travel down the geodesics. To this end we consider a one-parameter family of geodesics, $\gamma_s(t)$. That is, for each $s \in \mathbf{R}$, γ_s is a geodesic parameterized by the affine parameter t. The collection of these curves defines a smooth two-dimensional surface (embedded in a manifold M of arbitrary dimensionality). The coordinates on this surface may be chosen to be s and t, provided we have chosen a family of geodesics that do not cross. The entire surface is the set of points $x^\mu(s, t) \in M$. We have two natural vector fields: the tangent vectors to the geodesics,

$$T^\mu = \frac{\partial x^\mu}{\partial t}, \tag{3.201}$$

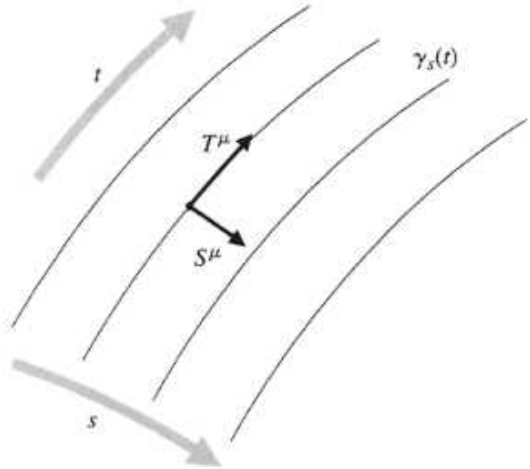


FIGURE 3.9 A set of geodesics $\gamma_s(t)$, with tangent vectors T^μ. The vector field S^μ measures the deviation between nearby geodesics.

and the deviation vectors

$$S^\mu = \frac{\partial x^\mu}{\partial s}. \tag{3.202}$$

This name derives from the informal notion that S^μ points from one geodesic toward the neighboring ones.

The idea that S^μ points from one geodesic to the next inspires us to define the "relative velocity of geodesics,"

$$V^\mu = (\nabla_T S)^\mu = T^\rho \nabla_\rho S^\mu, \tag{3.203}$$

and the "relative acceleration of geodesics,"

$$A^\mu = (\nabla_T V)^\mu = T^\rho \nabla_\rho V^\mu. \tag{3.204}$$

You should take the names with a grain of salt, but these vectors are certainly well-defined. This notion of *relative* acceleration between geodesics should be distinguished from the acceleration of a path away from being a geodesic, which would be given (when t is the proper time) by $a^\mu = T^\sigma \nabla_\sigma T^\mu$.

Since S and T are basis vectors adapted to a coordinate system, their commutator vanishes:

$$[S, T] = 0. \tag{3.205}$$

From (3.37) we then have

$$S^\rho \nabla_\rho T^\mu = T^\rho \nabla_\rho S^\mu. \tag{3.206}$$

With this in mind, let's compute the acceleration:

$$\begin{aligned} A^\mu &= T^\rho \nabla_\rho (T^\sigma \nabla_\sigma S^\mu) \\ &= T^\rho \nabla_\rho (S^\sigma \nabla_\sigma T^\mu) \\ &= (T^\rho \nabla_\rho S^\sigma)(\nabla_\sigma T^\mu) + T^\rho S^\sigma \nabla_\rho \nabla_\sigma T^\mu \\ &= (S^\rho \nabla_\rho T^\sigma)(\nabla_\sigma T^\mu) + T^\rho S^\sigma (\nabla_\sigma \nabla_\rho T^\mu + R^\mu{}_{\nu\rho\sigma} T^\nu) \\ &= (S^\rho \nabla_\rho T^\sigma)(\nabla_\sigma T^\mu) + S^\sigma \nabla_\sigma (T^\rho \nabla_\rho T^\mu) - (S^\sigma \nabla_\sigma T^\rho)\nabla_\rho T^\mu \\ &\quad + R^\mu{}_{\nu\rho\sigma} T^\nu T^\rho S^\sigma \\ &= R^\mu{}_{\nu\rho\sigma} T^\nu T^\rho S^\sigma . \end{aligned} \tag{3.207}$$

Let's think about this line by line. The first line is the definition of A^μ, and the second line comes directly from (3.206). The third line is simply the Leibniz rule. The fourth line replaces a double covariant derivative by the derivatives in the opposite order plus the Riemann tensor. In the fifth line we use Leibniz again (in the opposite order from usual), and then we cancel two identical terms and notice that the term involving $T^\rho \nabla_\rho T^\mu$ vanishes because T^μ is the tangent vector to a geodesic. The result,

$$\boxed{A^\mu = \frac{D^2}{dt^2} S^\mu = R^\mu{}_{\nu\rho\sigma} T^\nu T^\rho S^\sigma ,} \tag{3.208}$$

is the **geodesic deviation equation**. It expresses something that we might have expected: the relative acceleration between two neighboring geodesics is proportional to the curvature.

The geodesic deviation equation characterizes the behavior of a one-parameter family of neighboring geodesics. We will sometimes be interested in keeping track of the behavior of a multi-dimensional set of neighboring geodesics, perhaps representing a bundle of photons or a distribution of massive test particles. Such a set of geodesics forms a congruence; in Appendix F we derive equations that describe the evolution of such congruences.

Physically, of course, the acceleration of neighboring geodesics is interpreted as a manifestation of gravitational tidal forces. In the next chapter we explore in more detail how properties of curved spacetime are reflected by physics in a gravitational field.

3.11 ■ EXERCISES

1. Verify these consequences of metric compatibility ($\nabla_\sigma g_{\mu\nu} = 0$):

$$\begin{aligned} \nabla_\sigma g^{\mu\nu} &= 0 \\ \nabla_\lambda \epsilon_{\mu\nu\rho\sigma} &= 0. \end{aligned} \tag{3.209}$$

2. You are familiar with the operations of gradient ($\nabla\phi$), divergence ($\nabla \cdot \mathbf{V}$) and curl ($\nabla \times \mathbf{V}$) in ordinary vector analysis in three-dimensional Euclidean space. Using covariant derivatives, derive formulae for these operations in spherical polar coordinates $\{r, \theta, \phi\}$ defined by

$$x = r \sin\theta \cos\phi \tag{3.210}$$

$$y = r \sin\theta \sin\phi \tag{3.211}$$

$$z = r \cos\theta. \tag{3.212}$$

Compare your results to those in Jackson (1999) or an equivalent text. Are they identical? Should they be?

3. Imagine we have a *diagonal* metric $g_{\mu\nu}$. Show that the Christoffel symbols are given by

$$\Gamma^{\lambda}_{\mu\nu} = 0 \tag{3.213}$$

$$\Gamma^{\lambda}_{\mu\mu} = -\tfrac{1}{2}(g_{\lambda\lambda})^{-1}\partial_{\lambda} g_{\mu\mu} \tag{3.214}$$

$$\Gamma^{\lambda}_{\mu\lambda} = \partial_{\mu}\left(\ln\sqrt{|g_{\lambda\lambda}|}\right) \tag{3.215}$$

$$\Gamma^{\lambda}_{\lambda\lambda} = \partial_{\lambda}\left(\ln\sqrt{|g_{\lambda\lambda}|}\right) \tag{3.216}$$

In these expressions, $\mu \neq \nu \neq \lambda$, and repeated indices are *not* summed over.

4. In Euclidean three-space, we can define paraboloidal coordinates (u, v, ϕ) via

$$x = uv\cos\phi \quad y = uv\sin\phi \quad z = \tfrac{1}{2}(u^2 - v^2).$$

   (a) Find the coordinate transformation matrix between paraboloidal and Cartesian coordinates $\partial x^{\alpha}/\partial x^{\beta'}$ and the inverse transformation. Are there any singular points in the map?
   (b) Find the basis vectors and basis one-forms in terms of Cartesian basis vectors and forms.
   (c) Find the metric and inverse metric in paraboloidal coordinates.
   (d) Calculate the Christoffel symbols.
   (e) Calculate the divergence $\nabla_{\mu} V^{\mu}$ and Laplacian $\nabla_{\mu}\nabla^{\mu} f$.

5. Consider a 2-sphere with coordinates (θ, ϕ) and metric

$$ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2. \tag{3.217}$$

   (a) Show that lines of constant longitude (ϕ = constant) are geodesics, and that the only line of constant latitude (θ = constant) that is a geodesic is the equator ($\theta = \pi/2$).
   (b) Take a vector with components $V^{\mu} = (1, 0)$ and parallel-transport it once around a circle of constant latitude. What are the components of the resulting vector, as a function of θ?

6. A good approximation to the metric outside the surface of the Earth is provided by

$$ds^2 = -(1 + 2\Phi)\mathrm{d}t^2 + (1 - 2\Phi)\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2), \tag{3.218}$$

where

$$\Phi = -\frac{GM}{r} \tag{3.219}$$

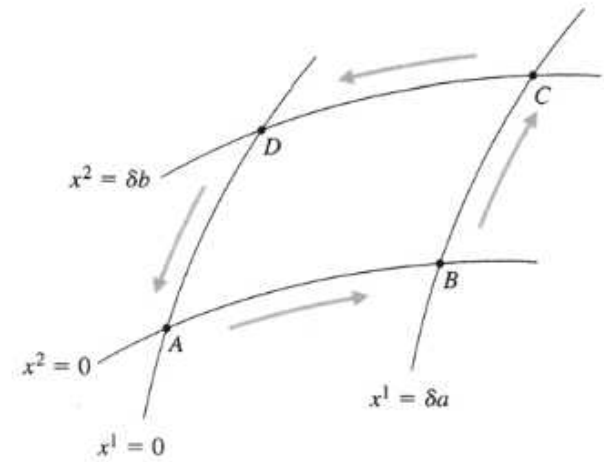
may be thought of as the familiar Newtonian gravitational potential. Here G is Newton's constant and M is the mass of the earth. For this problem Φ may be assumed to be small.

(a) Imagine a clock on the surface of the Earth at distance R_1 from the Earth's center, and another clock on a tall building at distance R_2 from the Earth's center. Calculate the time elapsed on each clock as a function of the coordinate time t. Which clock moves faster?

(b) Solve for a geodesic corresponding to a circular orbit around the equator of the Earth ($\theta = \pi/2$). What is $d\phi/dt$?

(c) How much proper time elapses while a satellite at radius R_1 (skimming along the surface of the earth, neglecting air resistance) completes one orbit? You can work to first order in Φ if you like. Plug in the actual numbers for the radius of the Earth and so on (don't forget to restore the speed of light) to get an answer in seconds. How does this number compare to the proper time elapsed on the clock stationary on the surface?

7. For this problem you will use the parallel propagator introduced in Appendix I to see how the Riemann tensor arises from parallel transport around an infinitesimal loop. Consider the following loop:



Using the infinite series expression for the parallel propagator, compute to lowest nontrivial order in δa and δb the transformation induced on a vector that is parallel transported around this loop from A to B to C to D and back to A, and show it is proportional to the appropriate components of the Riemann tensor. To make things easy, you can use x^1 and x^2 as parameters on the appropriate legs of the journey.

8. The metric for the three-sphere in coordinates $x^\mu = (\psi, \theta, \phi)$ can be written

$$ds^2 = d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{3.220}$$

(a) Calculate the Christoffel connection coefficients. Use whatever method you like, but it is good practice to get the connection coefficients by varying the integral (3.49).

(b) Calculate the Riemann tensor, Ricci tensor, and Ricci scalar.

(c) Show that (3.191) is obeyed by this metric, confirming that the three-sphere is a maximally symmetric space (as you would expect).

9. Show that the Weyl tensor $C^{\mu}{}_{\nu\rho\sigma}$ is left invariant by a conformal transformation.

10. Show that, for $n \geq 4$, the Weyl tensor satisfies a version of the Bianchi identity,

$$\nabla_{\rho} C^{\rho}{}_{\sigma\mu\nu} = 2\frac{(n-3)}{(n-2)}\left(\nabla_{[\mu} R_{\nu]\sigma} + \frac{1}{2(n-1)} g_{\sigma[\mu}\nabla_{\nu]} R\right). \tag{3.221}$$

11. Since the Poincare half-plane with metric (3.192) is maximally symmetric, we might expect that it is rotationally symmetric around any point, although this certainly isn't evident in the $\{x, y\}$ coordinates. If that is so, it should be possible to put the metric in a form where the rotational symmetry is manifest, such as

$$ds^2 = f^2(r)[\mathrm{d}r^2 + r^2 \mathrm{d}\theta^2]. \tag{3.222}$$

To show that this works, calculate the curvature scalar for this metric and solve for the function $f(r)$ subject to the condition $R = -2/a^2$ everywhere. What is the range of the coordinate r?

12. Show that any Killing vector K^{μ} satisfies the relations mentioned in the text:

$$\nabla_{\mu}\nabla_{\sigma} K^{\rho} = R^{\rho}{}_{\sigma\mu\nu} K^{\nu}$$

$$K^{\lambda}\nabla_{\lambda} R = 0. \tag{3.223}$$

13. Find explicit expressions for a complete set of Killing vector fields for the following spaces:

(a) Minkowski space, with metric $ds^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$.

(b) A spacetime with coordinates $\{u, v, x, y\}$ and metric

$$ds^2 = -(\mathrm{d}u\mathrm{d}v + \mathrm{d}v\mathrm{d}u) + a^2(u)\mathrm{d}x^2 + b^2(u)\mathrm{d}y^2, \tag{3.224}$$

where a and b are unspecified functions of u. This represents a gravitational wave spacetime. (*Hints*, which you need not show: there are five Killing vectors in all, and all of them have a vanishing u component K^u.)

Be careful, in all of these cases, about the distinction between upper and lower indices.

14. Consider the three Killing vectors of the the two-sphere, (3.188). Show that their commutators satisfy the following algebra:

$$[R, S] = T$$

$$[S, T] = R$$

$$[T, R] = S. \tag{3.225}$$

15. Use Raychaudhuri's equation, discussed in Appendix F, to show that, if a fluid is flowing on geodesics through spacetime with zero shear and expansion, then spacetime must have a timelike Killing vector.

16. Consider again the metric on a three-sphere,

$$ds^2 = \mathrm{d}\psi^2 + \sin^2\psi(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2). \tag{3.226}$$

In this problem we make use of noncoordinate bases, discussed in Appendix J. In an orthonormal frame of one-forms $\hat{\theta}^{(a)}$ the metric would become

$$ds^2 = \hat{\theta}^{(1)} \otimes \hat{\theta}^{(1)} + \hat{\theta}^{(2)} \otimes \hat{\theta}^{(2)} + \hat{\theta}^{(3)} \otimes \hat{\theta}^{(3)}. \tag{3.227}$$

(a) Find such an orthonormal frame of one-forms, such that the matrix e_a^μ is diagonal. Don't worry about covering the entire manifold.

(b) Compute the components of the spin connection by solving $\mathrm{d}e^a + \omega^a{}_b \wedge e^b = 0$.

(c) Compute the components of the Riemann tensor $R^\rho{}_{\sigma\mu\nu}$ in the coordinate basis adapted to x^μ by computing the components of the curvature two-form $R^a{}_{b\mu\nu}$ and then converting.

CHAPTER

4 Gravitation

4.1 ■ PHYSICS IN CURVED SPACETIME

Having paid our mathematical dues, we are now prepared to examine the physics of gravitation as described by general relativity. This subject falls naturally into two pieces: how the gravitational field influences the behavior of matter, and how matter determines the gravitational field. In Newtonian gravity, these two elements consist of the expression for the acceleration of a body in a gravitational potential Φ,

$$\mathbf{a} = -\nabla\Phi, \tag{4.1}$$

and Poisson's differential equation for the potential in terms of the matter density ρ and Newton's gravitational constant G:

$$\nabla^2\Phi = 4\pi G\rho. \tag{4.2}$$

In general relativity, the analogous statements will describe how the curvature of spacetime acts on matter to manifest itself as gravity, and how energy and momentum influence spacetime to create curvature. In either case it would be legitimate to start at the top, by stating outright the laws governing physics in curved spacetime and working out their consequences. Instead, we will try to be a little more motivational, starting with basic physical principles and attempting to argue that these lead naturally to an almost unique physical theory.

In Chapter 2 we motivated our discussion of manifolds by introducing the Einstein Equivalence Principle, or EEP: "In small enough regions of spacetime, the laws of physics reduce to those of special relativity; it is impossible to detect the existence of a gravitational field by means of local experiments." The EEP arises from the idea that gravity is *universal*; it affects all particles (and indeed all forms of energy-momentum) in the same way. This feature of universality led Einstein to propose that what we experience as gravity is a manifestation of the curvature of spacetime. The idea is simply that something so universal as gravitation could be most easily described as a fundamental feature of the background on which matter fields propagate, as opposed to as a conventional force. At the same time, the identification of spacetime as a curved manifold is supported by the similarity between the undetectability of gravity in local regions and our ability to find locally inertial coordinates ($g_{\hat{\mu}\hat{\nu}} = \eta_{\hat{\mu}\hat{\nu}}$, $\partial_{\hat{\rho}} g_{\hat{\mu}\hat{\nu}} = 0$ at a point p) on a manifold.

Best of all, this abstract philosophizing translates directly into a simple recipe for generalizing laws of physics to the curved-spacetime context, known as the **minimal-coupling principle**. In its baldest form, this recipe may be stated as follows:

1. Take a law of physics, valid in inertial coordinates in flat spacetime.
2. Write it in a coordinate-invariant (tensorial) form.
3. Assert that the resulting law remains true in curved spacetime.

It may seem somewhat melodramatic to take such a simple idea and spread it out into a three-part procedure. We hope only to make clear that there is nothing very complicated going on. Operationally, this recipe usually amounts to taking an agreed-upon law in flat space and replacing the Minkowski metric $\eta_{\mu\nu}$ by the more general metric $g_{\mu\nu}$, and replacing partial derivatives ∂_μ by covariant derivatives ∇_μ. For this reason, this recipe is sometimes known as the "Comma-Goes-to-Semicolon Rule," by those who use commas and semicolons to denote partial and covariant derivatives.

As a straightforward example, we can consider the motion of freely-falling (unaccelerated) particles. In flat space such particles move in straight lines; in equations, this is expressed as the vanishing of the second derivative of the parameterized path $x^\mu(\lambda)$:

$$\frac{d^2x^\mu}{d\lambda^2} = 0. \tag{4.3}$$

This is not, in general coordinates, a tensorial equation; although $dx^\mu/d\lambda$ are the components of a well-defined vector, the second derivative components $d^2x^\mu/d\lambda^2$ are not. You might really think that this is a tensorial-looking equation; however, you can readily check that it's not even true in polar coordinates, unless you expect free particles to move in circles. We can use the chain rule to write

$$\frac{d^2x^\mu}{d\lambda^2} = \frac{dx^\nu}{d\lambda}\partial_\nu\frac{dx^\mu}{d\lambda}. \tag{4.4}$$

Now it is clear how to generalize this to curved space—simply replace the partial derivative by a covariant one,

$$\frac{dx^\nu}{d\lambda}\partial_\nu\frac{dx^\mu}{d\lambda} \quad\rightarrow\quad \frac{dx^\nu}{d\lambda}\nabla_\nu\frac{dx^\mu}{d\lambda} = \frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda}. \tag{4.5}$$

We recognize, then, that the appropriate general-relativistic version of the Newtonian relation (4.3) is simply the geodesic equation,

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0. \tag{4.6}$$

In general relativity, therefore, free particles move along geodesics; we have mentioned this before, but now you have a slightly better idea why it is true.

As an even more straightforward example, and one that we have referred to already, we have the law of energy-momentum conservation in flat spacetime:

$$\partial_\mu T^{\mu\nu} = 0. \tag{4.7}$$

Plugging into our recipe reveals the appropriate generalization to curved spacetime:

$$\nabla_\mu T^{\mu\nu} = 0. \tag{4.8}$$

It really is just that simple—sufficiently so that we felt quite comfortable using this equation in Chapter 3, without any detailed justification. Of course, this simplicity should not detract from the profound consequences of the generalization to curved spacetime, as illustrated in the example of the expanding universe.

It is one thing to generalize an equation from flat to curved spacetime; it is something altogether different to argue that the result describes gravity. To do so, we can show how the usual results of Newtonian gravity fit into the picture. We define the Newtonian limit by three requirements: the particles are moving slowly (with respect to the speed of light), the gravitational field is weak (so that it can be considered as a perturbation of flat space), and the field is also static (unchanging with time). Let us see what these assumptions do to the geodesic equation, taking the proper time τ as an affine parameter. "Moving slowly" means that

$$\frac{dx^i}{d\tau} \ll \frac{dt}{d\tau}, \tag{4.9}$$

so the geodesic equation becomes

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}\left(\frac{dt}{d\tau}\right)^2 = 0. \tag{4.10}$$

Since the field is static ($\partial_0 g_{\mu\nu} = 0$), the relevant Christoffel symbols Γ^μ_{00} simplify:

$$\begin{aligned}\Gamma^\mu_{00} &= \tfrac{1}{2}g^{\mu\lambda}(\partial_0 g_{\lambda 0} + \partial_0 g_{0\lambda} - \partial_\lambda g_{00}) \\ &= -\tfrac{1}{2}g^{\mu\lambda}\partial_\lambda g_{00}.\end{aligned} \tag{4.11}$$

Finally, the weakness of the gravitational field allows us to decompose the metric into the Minkowski form plus a small perturbation:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1. \tag{4.12}$$

We are working in inertial coordinates, so $\eta_{\mu\nu}$ is the canonical form of the metric. The "smallness condition" on the metric perturbation $h_{\mu\nu}$ doesn't really make sense in arbitrary coordinates. From the definition of the inverse metric, $g^{\mu\nu}g_{\nu\sigma} = \delta^\mu_\sigma$, we find that to first order in h,

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}, \tag{4.13}$$

where $h^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma}h_{\rho\sigma}$. In fact, we can use the Minkowski metric to raise and lower indices on an object of any definite order in h, since the corrections would only contribute at higher orders. If you like, think of $h_{\mu\nu}$ as a symmetric (0, 2) tensor field propagating in Minkowski space and interacting with other fields.

Putting it all together, to first order in $h_{\mu\nu}$ we find

$$\Gamma^{\mu}_{00} = -\tfrac{1}{2}\eta^{\mu\lambda}\partial_\lambda h_{00}. \tag{4.14}$$

The geodesic equation (4.10) is therefore

$$\frac{d^2x^\mu}{d\tau^2} = \frac{1}{2}\eta^{\mu\lambda}\partial_\lambda h_{00}\left(\frac{dt}{d\tau}\right)^2. \tag{4.15}$$

Using $\partial_0 h_{00} = 0$, the $\mu = 0$ component of this is just

$$\frac{d^2t}{d\tau^2} = 0. \tag{4.16}$$

That is, $dt/d\tau$ is constant. To examine the spacelike components of (4.15), recall that the spacelike components of $\eta^{\mu\nu}$ are just those of a 3×3 identity matrix. We therefore have

$$\frac{d^2x^i}{d\tau^2} = \frac{1}{2}\left(\frac{dt}{d\tau}\right)^2\partial_i h_{00}. \tag{4.17}$$

Dividing both sides by $(dt/d\tau)^2$ has the effect of converting the derivative on the left-hand side from τ to t, leaving us with

$$\frac{d^2x^i}{dt^2} = \frac{1}{2}\partial_i h_{00}. \tag{4.18}$$

This begins to look a great deal like Newton's theory of gravitation. In fact, if we compare this equation to (4.1), we find that they are the same once we identify

$$h_{00} = -2\Phi, \tag{4.19}$$

or in other words

$$g_{00} = -(1 + 2\Phi). \tag{4.20}$$

Therefore, we have shown that the curvature of spacetime is indeed sufficient to describe gravity in the Newtonian limit, as long as the metric takes the form (4.20). It remains, of course, to find field equations for the metric that imply this is the form taken, and that for a single gravitating body we recover the Newtonian formula

$$\Phi = -\frac{GM}{r}, \tag{4.21}$$

but that will come soon enough.

The straightforward procedure we have outlined for generalizing laws of physics to curved spacetime does have some subtleties, which we address in Section 4.7. But it's more than good enough for our present purposes, so let's not delay our pursuit of the second half of our task, obtaining the field equation for the metric in general relativity.

4.2 ■ EINSTEIN'S EQUATION

Just as Maxwell's equations govern how the electric and magnetic fields respond to charges and currents, Einstein's field equation governs how the metric responds to energy and momentum. Ultimately the field equation must be postulated and tested against experiment, not derived from any bedrock principles; however, we can motivate it on the basis of plausibility arguments. We will actually do this in two ways: first by some informal reasoning by analogy, close to what Einstein himself was thinking, and then by starting with an action and deriving the corresponding equations of motion.

The informal argument begins with the realization that we would like to find an equation that supersedes the Poisson equation for the Newtonian potential:

$$\nabla^2\Phi = 4\pi G\rho, \tag{4.22}$$

where $\nabla^2 = \delta^{ij}\partial_i\partial_j$ is the Laplacian in space and ρ is the mass density. [The explicit form of Φ given in (4.21) is one solution of (4.22), for the case of a pointlike mass distribution.] What characteristics should our sought-after equation possess? On the left-hand side of (4.22) we have a second-order differential operator acting on the gravitational potential, and on the right-hand side a measure of the mass distribution. A relativistic generalization should take the form of an equation between tensors. We know what the tensor generalization of the mass density is; it's the energy-momentum tensor $T_{\mu\nu}$. The gravitational potential, meanwhile, should get replaced by the metric tensor, because in (4.20) we had to relate a perturbation of the metric to the Newtonian potential to successfully reproduce gravity. We might therefore guess that our new equation will have $T_{\mu\nu}$ set proportional to some tensor, which is second-order in derivatives of the metric; something along the lines of

$$[\nabla^2 g]_{\mu\nu} \propto T_{\mu\nu}, \tag{4.23}$$

but of course we want it to be completely tensorial.

The left-hand side of (4.23) is not a sensible tensor; it's just a suggestive notation to indicate that we would like a symmetric (0, 2) tensor that is second-order in derivatives of the metric. The first choice might be to act the d'Alembertian $\Box = \nabla^\mu\nabla_\mu$ on the metric $g_{\mu\nu}$, but this is automatically zero by metric compatibility. Fortunately, there is an obvious quantity which is not zero and is constructed from second derivatives (and first derivatives) of the metric: the Riemann tensor $R^\rho{}_{\sigma\mu\nu}$. Recall that the Riemann tensor is constructed from the Christoffel sym-

bols and their first derivatives, and the Christoffel symbols are constructed from the metric and its first derivatives, so $R^{\rho}{}_{\sigma\mu\nu}$ contains second derivatives of $g_{\mu\nu}$. It doesn't have the right number of indices, but we can contract it to form the Ricci tensor $R_{\mu\nu}$, which does (and is symmetric to boot). It is therefore tempting to guess that the gravitational field equations are

$$R_{\mu\nu} = \kappa T_{\mu\nu}, \tag{4.24}$$

for some constant κ. In fact, Einstein did suggest this equation at one point. There is a problem, unfortunately, with conservation of energy. If we want to preserve

$$\nabla^{\mu} T_{\mu\nu} = 0, \tag{4.25}$$

by (4.24) we would have

$$\nabla^{\mu} R_{\mu\nu} = 0. \tag{4.26}$$

This is certainly not true in an arbitrary geometry; we have seen from the Bianchi identity (3.150) that

$$\nabla^{\mu} R_{\mu\nu} = \tfrac{1}{2} \nabla_{\nu} R. \tag{4.27}$$

But our proposed field equation implies that $R = \kappa g^{\mu\nu} T_{\mu\nu} = \kappa T$, so taking these together we have

$$\nabla_{\mu} T = 0. \tag{4.28}$$

The covariant derivative of a scalar is just the partial derivative, so (4.28) is telling us that T is constant throughout spacetime. This is highly implausible, since $T = 0$ in vacuum while $T \neq 0$ in matter. We have to try harder.

Of course we don't have to try much harder, since we already know of a symmetric $(0, 2)$ tensor, constructed from the Ricci tensor, which is automatically conserved: the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu}, \tag{4.29}$$

which always obeys $\nabla^{\mu} G_{\mu\nu} = 0$. We are therefore led to propose

$$G_{\mu\nu} = \kappa T_{\mu\nu} \tag{4.30}$$

as a field equation for the metric. (Actually it is probably more common to write out $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$, rather than use the abbreviation $G_{\mu\nu}$.) This equation satisfies all of the obvious requirements: the right-hand side is a covariant expression of the energy and momentum density in the form of a symmetric and conserved $(0, 2)$ tensor, while the left-hand side is a symmetric and conserved $(0, 2)$ tensor constructed from the metric and its first and second derivatives. It only remains to fix the proportionality constant κ, and to see whether the result actually repro-

duces gravity as we know it. In other words, does this equation predict the Poisson equation for the gravitational potential in the Newtonian limit?

To answer this, note that contracting both sides of (4.30) yields (in four dimensions)

$$R = -\kappa T, \tag{4.31}$$

and using this we can rewrite (4.30) as

$$R_{\mu\nu} = \kappa(T_{\mu\nu} - \tfrac{1}{2}T g_{\mu\nu}). \tag{4.32}$$

This is the same equation, just written slightly differently. We would like to see if it predicts Newtonian gravity in the weak-field, time-independent, slowly-moving-particles limit. We consider a perfect-fluid source of energy-momentum, for which

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu}, \tag{4.33}$$

where U^μ is the fluid four-velocity and ρ and p are the rest-frame energy and momentum densities. In fact for the Newtonian limit we may neglect the pressure; roughly speaking, the pressure of a body becomes important when its constituent particles are traveling at speeds close to that of light, which we exclude from the Newtonian limit by hypothesis. So we are actually considering the energy-momentum tensor of dust:

$$T_{\mu\nu} = \rho U_\mu U_\nu. \tag{4.34}$$

The "fluid" we are considering is some massive body, such as the Earth or the Sun. We will work in the fluid rest frame, in which

$$U^\mu = (U^0, 0, 0, 0). \tag{4.35}$$

The timelike component can be fixed by appealing to the normalization condition $g_{\mu\nu}U^\mu U^\nu = -1$. In the weak-field limit we write, in accordance with (4.12) and (4.13),

$$\begin{aligned} g_{00} &= -1 + h_{00}, \\ g^{00} &= -1 - h_{00}. \end{aligned} \tag{4.36}$$

Then to first order in $h_{\mu\nu}$ we get

$$U^0 = 1 + \tfrac{1}{2}h_{00}. \tag{4.37}$$

In fact, however, this is needlessly careful, as we are going to plug the four-velocity into (4.34), and the energy density ρ is already considered small (spacetime will be flat as ρ is taken to zero). So to our level of approximation, we can simply take $U^0 = 1$, and correspondingly $U_0 = -1$. Then

$$T_{00} = \rho, \tag{4.38}$$

and all other components vanish. In this limit the rest energy $\rho = T_{00}$ will be much larger than the other terms in $T_{\mu\nu}$, so we want to focus on the $\mu = 0$, $\nu = 0$ component of (4.32). The trace, to lowest nontrivial order, is

$$T = g^{00}T_{00} = -T_{00} = -\rho. \tag{4.39}$$

We plug this into the 00 component of our proposed gravitational field equation (4.32), to get

$$R_{00} = \tfrac{1}{2}\kappa\rho. \tag{4.40}$$

This is an equation relating derivatives of the metric to the energy density. To find the explicit expression in terms of the metric, we need to evaluate $R_{00} = R^{\lambda}{}_{0\lambda 0}$. In fact we only need $R^{i}{}_{0i0}$, since $R^{0}{}_{000} = 0$. We have

$$R^{i}{}_{0j0} = \partial_j \Gamma^{i}_{00} - \partial_0 \Gamma^{i}_{j0} + \Gamma^{i}_{j\lambda}\Gamma^{\lambda}_{00} - \Gamma^{i}_{0\lambda}\Gamma^{\lambda}_{j0}. \tag{4.41}$$

The second term here is a time derivative, which vanishes for static fields. The third and fourth terms are of the form $(\Gamma)^2$, and since Γ is first-order in the metric perturbation these contribute only at second order, and can be neglected. We are left with $R^{i}{}_{0j0} = \partial_j \Gamma^{i}_{00}$. From this we get

$$\begin{aligned} R_{00} &= R^{i}{}_{0i0} \\ &= \partial_i \left[\tfrac{1}{2} g^{i\lambda} (\partial_0 g_{\lambda 0} + \partial_0 g_{0\lambda} - \partial_\lambda g_{00}) \right] \\ &= -\tfrac{1}{2}\delta^{ij}\partial_i \partial_j h_{00} \\ &= -\tfrac{1}{2}\nabla^2 h_{00}. \end{aligned} \tag{4.42}$$

Comparing to (4.40), we see that the 00 component of (4.30) in the Newtonian limit predicts

$$\nabla^2 h_{00} = -\kappa\rho. \tag{4.43}$$

Since (4.19) sets $h_{00} = -2\Phi$, this is precisely the Poisson equation (4.22), if we set $\kappa = 8\pi G$.

So our guess, (4.30), seems to have worked out. With the normalization chosen so as to correctly recover the Newtonian limit, we can present **Einstein's equation** for general relativity:

$$\boxed{R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}.} \tag{4.44}$$

This tells us how the curvature of spacetime reacts to the presence of energy-momentum. G is of course Newton's constant of gravitation; it has nothing to do with the trace of $G_{\mu\nu}$. Einstein, you may have heard, thought that the left-hand side was nice and geometrical, while the right-hand side was somewhat less compelling.

It is sometimes useful to rewrite Einstein's equation in a slightly different form. Following (4.31) and (4.32), we can take the trace of (4.44) to find that $R = -8\pi GT$. Plugging this in and moving the trace term to the right-hand side, we obtain

$$\boxed{R_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \tfrac{1}{2}Tg_{\mu\nu}\right).} \tag{4.45}$$

The difference between this and (4.44) is purely cosmetic; in substance they are precisely the same. We will often be interested in the Einstein's equation in vacuum, where $T_{\mu\nu} = 0$ (for example, outside a star or planet). Then of course the right-hand side of (4.45) vanishes. Therefore the vacuum Einstein equation is simply

$$\boxed{R_{\mu\nu} = 0.} \tag{4.46}$$

This is both slightly less formidable, and of considerable physical usefulness.

4.3 ■ LAGRANGIAN FORMULATION

An alternative route to Einstein's equation is through the principle of least action, as we discussed for classical field theories in flat spacetime at the end of Chapter 1. Let's spend a moment to generalize those results to curved spacetime, and then see what kind of Lagrangian is appropriate for general relativity. We'll work in n dimensions, since our results will not depend on the dimensionality; we will, however, assume that our metric has Lorentzian signature.

Consider a field theory in which the dynamical variables are a set of fields $\Phi^i(x)$. The classical solutions to such a theory will be those that are critical points of an action S, generally expressed as an integral over space of a Lagrange density $\mathcal{L}$,

$$S = \int \mathcal{L}(\Phi^i, \nabla_\mu \Phi^i)\, d^n x. \tag{4.47}$$

Note that we are now imagining that the Lagrangian is a function of the fields and their covariant (rather than partial) derivatives, as is appropriate in curved space. Note also that, since $d^n x$ is a density rather than a tensor, $\mathcal{L}$ is also a density (since their product must be a well-defined tensor); we typically write

$$\mathcal{L} = \sqrt{-g}\,\widehat{\mathcal{L}}, \tag{4.48}$$

where $\widehat{\mathcal{L}}$ is indeed a scalar. You might think it would be sensible to forget about what we are calling $\mathcal{L}$ and just focus on $\widehat{\mathcal{L}}$, but in fact both quantities are useful in different circumstances; it is $\mathcal{L}$ that will matter whenever we are varying with

respect to the metric itself. The associated Euler–Lagrange equations make use of the scalar $\widehat{\mathcal{L}}$, and are otherwise like those in flat space, but with covariant instead of partial derivatives:

$$\frac{\partial \widehat{\mathcal{L}}}{\partial \Phi} - \nabla_\mu \left(\frac{\partial \widehat{\mathcal{L}}}{\partial (\nabla_\mu \Phi)} \right) = 0. \tag{4.49}$$

In deriving these equations, we make use of Stokes's theorem (3.35),

$$\int_\Sigma \nabla_\mu V^\mu \sqrt{|g|}\, d^n x = \int_{\partial\Sigma} n_\mu V^\mu \sqrt{|\gamma|}\, d^{n-1} x, \tag{4.50}$$

and set the variation equal to zero at infinity (the boundary). Integration by parts therefore takes the form

$$\int A^\mu (\nabla_\mu B) \sqrt{-g}\, d^n x = -\int (\nabla_\mu A^\mu) B \sqrt{-g}\, d^n x + \text{boundary terms}. \tag{4.51}$$

For example, the curved-spacetime generalization of the action for a single scalar field ϕ considered in Chapter 1 would be

$$S_\phi = \int \left[-\frac{1}{2} g^{\mu\nu} (\nabla_\mu \phi)(\nabla_\nu \phi) - V(\phi) \right] \sqrt{-g}\, d^n x, \tag{4.52}$$

which would lead to an equation of motion

$$\Box \phi - \frac{dV}{d\phi} = 0, \tag{4.53}$$

where the covariant d'Alembertian is

$$\Box = \nabla^\mu \nabla_\mu = g^{\mu\nu} \nabla_\mu \nabla_\nu. \tag{4.54}$$

Just as in flat spacetime, the combination $g^{\mu\nu}(\nabla_\mu \phi)(\nabla_\nu \phi)$ is often abbreviated as $(\nabla\phi)^2$. Of course, the covariant derivatives are equivalent to partial derivatives when acting on scalars, but it is wise to use the ∇_μ notation still; you never know when you might integrate by parts and suddenly be acting on a vector.

With that as a warm-up, we turn to the construction of an action for general relativity. Our dynamical variable is now the metric $g_{\mu\nu}$; what scalars can we make out of the metric to serve as a Lagrangian? Since we know that the metric can be set equal to its canonical form and its first derivatives set to zero at any one point, any nontrivial scalar must involve at least second derivatives of the metric. The Riemann tensor is of course made from second derivatives of the metric, and we argued earlier that the only independent scalar we could construct from the Riemann tensor was the Ricci scalar R. What we did not show, but is nevertheless true, is that any nontrivial tensor made from products of the metric and its first and second derivatives can be expressed in terms of the metric and the Riemann tensor. Therefore, the *only* independent scalar constructed from the metric, which

is no higher than second order in its derivatives, is the Ricci scalar. Hilbert figured that this was therefore the simplest possible choice for a Lagrangian, and proposed

$$S_H = \int \sqrt{-g} R \, d^n x, \tag{4.55}$$

known as the **Hilbert action** (or sometimes the Einstein–Hilbert action). As we shall see, he was right.

The equation of motion should come from varying the action with respect to the metric. Unfortunately the action isn't quite in the form (4.47), since it can't be written in terms of covariant derivatives of $g_{\mu\nu}$ (which would simply vanish). Therefore, instead of simply plugging into the Euler–Lagrange equations, we will consider directly the behavior of S_H under small variations of the metric. In fact it is more convenient to vary with respect to the inverse metric $g^{\mu\nu}$. Since $g^{\mu\lambda} g_{\lambda\nu} = \delta^\mu_\nu$, and the Kronecker delta is unchanged under any variation, it is straightforward to express variations of the metric and inverse metric in terms of each other:

$$\delta g_{\mu\nu} = -g_{\mu\rho} g_{\nu\sigma} \delta g^{\rho\sigma}, \tag{4.56}$$

so stationary points with respect to variations in $g^{\mu\nu}$ are equivalent to those with respect to variations in $g_{\mu\nu}$. Using $R = g^{\mu\nu} R_{\mu\nu}$, we have

$$\delta S_H = (\delta S)_1 + (\delta S)_2 + (\delta S)_3, \tag{4.57}$$

where

$$\begin{aligned} (\delta S)_1 &= \int d^n x \, \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} \\ (\delta S)_2 &= \int d^n x \, \sqrt{-g} R_{\mu\nu} \delta g^{\mu\nu} \\ (\delta S)_3 &= \int d^n x \, R \delta \sqrt{-g}. \end{aligned} \tag{4.58}$$

The second term $(\delta S)_2$ is already in the form of some expression multiplied by $\delta g^{\mu\nu}$; let's examine the others more closely.

Recall that the Ricci tensor is the contraction of the Riemann tensor, which is given by

$$R^\rho{}_{\mu\lambda\nu} = \partial_\lambda \Gamma^\rho_{\nu\mu} + \Gamma^\rho_{\lambda\sigma} \Gamma^\sigma_{\nu\mu} - (\lambda \leftrightarrow \nu). \tag{4.59}$$

The variation of the Riemann tensor with respect to the metric can be found by first varying the connection with respect to the metric, and then substituting into this expression. However, let us consider arbitrary variations of the connection by replacing

$$\Gamma^\rho_{\nu\mu} \to \Gamma^\rho_{\nu\mu} + \delta\Gamma^\rho_{\nu\mu}. \tag{4.60}$$

The variation $\delta\Gamma^{\rho}_{\nu\mu}$ is the difference of two connections, and therefore is itself a tensor. We can thus take its covariant derivative,

$$\nabla_\lambda(\delta\Gamma^{\rho}_{\nu\mu}) = \partial_\lambda(\delta\Gamma^{\rho}_{\nu\mu}) + \Gamma^{\rho}_{\lambda\sigma}\delta\Gamma^{\sigma}_{\nu\mu} - \Gamma^{\sigma}_{\lambda\nu}\delta\Gamma^{\rho}_{\sigma\mu} - \Gamma^{\sigma}_{\lambda\mu}\delta\Gamma^{\rho}_{\nu\sigma}. \tag{4.61}$$

Here and elsewhere, the covariant derivatives are taken with respect to $g_{\mu\nu}$, not $g_{\mu\nu} + \delta g_{\mu\nu}$. Given this expression and a small amount of labor, it is easy to show that, to first order in the variation,

$$\delta R^{\rho}{}_{\mu\lambda\nu} = \nabla_\lambda(\delta\Gamma^{\rho}_{\nu\mu}) - \nabla_\nu(\delta\Gamma^{\rho}_{\lambda\mu}). \tag{4.62}$$

You are encouraged check this yourself. Therefore, the contribution of the first term in (4.58) to δS can be written

$$\begin{aligned}(\delta S)_1 &= \int d^n x\sqrt{-g}\, g^{\mu\nu}\left[\nabla_\lambda(\delta\Gamma^{\lambda}_{\nu\mu}) - \nabla_\nu(\delta\Gamma^{\lambda}_{\lambda\mu})\right] \\ &= \int d^n x\sqrt{-g}\,\nabla_\sigma\left[g^{\mu\nu}(\delta\Gamma^{\sigma}_{\mu\nu}) - g^{\mu\sigma}(\delta\Gamma^{\lambda}_{\lambda\mu})\right],\end{aligned} \tag{4.63}$$

where we have used metric compatibility and relabeled some dummy indices. We can now plug in the expression for $\delta\Gamma^{\sigma}_{\mu\nu}$ in terms of $\delta g^{\mu\nu}$, which works out to be

$$\delta\Gamma^{\sigma}_{\mu\nu} = -\tfrac{1}{2}\left[g_{\lambda\mu}\nabla_\nu(\delta g^{\lambda\sigma}) + g_{\lambda\nu}\nabla_\mu(\delta g^{\lambda\sigma}) - g_{\mu\alpha}g_{\nu\beta}\nabla^\sigma(\delta g^{\alpha\beta})\right], \tag{4.64}$$

leading to

$$(\delta S)_1 = \int d^n x\sqrt{-g}\,\nabla_\sigma[g_{\mu\nu}\nabla^\sigma(\delta g^{\mu\nu}) - \nabla_\lambda(\delta g^{\sigma\lambda})], \tag{4.65}$$

as you are also welcome to check. But (4.63) [or (4.65)] is an integral with respect to the natural volume element of the covariant divergence of a vector; by Stokes's theorem, this is equal to a boundary contribution at infinity, which we can set to zero by making the variation vanish at infinity. Therefore this term contributes nothing to the total variation. Although to be honest, we have cheated. The boundary term will include not only the metric variation, but also its first derivative, which is not traditionally set to zero. For our present purposes it doesn't matter, but in principle we might care about what happens at the boundary, and would have to include an additional term in the action to take care of this subtlety.

To make sense of the $(\delta S)_3$ term we need to use the following fact, true for any square matrix M with nonvanishing determinant:

$$\ln(\det M) = \mathrm{Tr}(\ln M). \tag{4.66}$$

Here, $\ln M$ is defined by $\exp(\ln M) = M$. For numbers this is obvious, for matrices it's a little less straightforward. The variation of this identity yields

$$\frac{1}{\det M}\delta(\det M) = \mathrm{Tr}(M^{-1}\delta M). \tag{4.67}$$

We have used the cyclic property of the trace to allow us to ignore the fact that M^{-1} and δM may not commute. Taking the matrix M to be the metric $g_{\mu\nu}$, so that $\det M = \det g_{\mu\nu} = g$, we get

$$\begin{aligned}\delta g &= g(g^{\mu\nu}\delta g_{\mu\nu}) \\ &= -g(g_{\mu\nu}\delta g^{\mu\nu}).\end{aligned} \tag{4.68}$$

In the last step we converted from $\delta g_{\mu\nu}$ to $\delta g^{\mu\nu}$ using (4.56). Now we can just plug in to get

$$\begin{aligned}\delta\sqrt{-g} &= -\frac{1}{2\sqrt{-g}}\delta g \\ &= \frac{1}{2}\frac{g}{\sqrt{-g}}g_{\mu\nu}\delta g^{\mu\nu} \\ &= -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}.\end{aligned} \tag{4.69}$$

Hearkening back to (4.58), and remembering that $(\delta S)_1$ does not contribute, we find

$$\delta S_H = \int d^n x\sqrt{-g}\left[R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right]\delta g^{\mu\nu}. \tag{4.70}$$

Recall that the functional derivative of the action satisfies

$$\delta S = \int \sum_i \left(\frac{\delta S}{\delta \Phi^i}\,\delta\Phi^i\right) d^n x, \tag{4.71}$$

where $\{\Phi^i\}$ is a complete set of fields being varied (in our case, it's just $g^{\mu\nu}$). Stationary points are those for which each $\delta S/\delta\Phi^i = 0$, so we recover Einstein's equation in vacuum:

$$\frac{1}{\sqrt{-g}}\frac{\delta S_H}{\delta g^{\mu\nu}} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0. \tag{4.72}$$

The advantage of the Lagrangian approach is manifested by the fact that our very first guess (which was practically unique) gave the right answer, in contrast with our previous trial-and-error method. This is a reflection of two elegant features of this technique: First, the Lagrangian is a scalar, rather than a tensor, and therefore more restricted; second, the symmetries of the theory are straightforwardly imposed (in this case, we automatically derived a tensor with vanishing divergence, which is related to diffeomorphism invariance, as discussed in Appendix B).

We derived Einstein's equation "in vacuum" because we only included the gravitational part of the action, not additional terms for matter fields. What we would really like, however, is to get the nonvacuum field equation as well. That

means we consider an action of the form

$$S = \frac{1}{16\pi G} S_H + S_M, \tag{4.73}$$

where S_M is the action for matter, and we have presciently normalized the gravitational action so that we get the right answer. Following through the same procedure as above leads to

$$\frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} = \frac{1}{16\pi G} \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) + \frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}} = 0. \tag{4.74}$$

We now boldly define the energy-momentum tensor to be

$$T_{\mu\nu} = -2 \frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}}. \tag{4.75}$$

This allows us to recover the complete Einstein's equation,

$$R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{4.76}$$

or equivalently, $G_{\mu\nu} = 8\pi G T_{\mu\nu}$.

Why should we think that (4.75) is really the energy-momentum tensor? In some sense it is only because it is a symmetric, conserved, (0, 2) tensor with dimensions of energy density; if you prefer to call it by some other name, go ahead. But it also accords with our preconceived expectations. Consider again the action for a scalar field, (4.52). Now vary this action with respect, not to ϕ, but to the inverse metric:

$$\delta S_\phi = \int d^n x \left[\sqrt{-g} \left(-\frac{1}{2} \delta g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi \right) + \delta \sqrt{-g} \left(-\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V(\phi) \right) \right] \tag{4.77}$$

$$= \int d^n x \sqrt{-g}\, \delta g^{\mu\nu} \left[-\frac{1}{2} \nabla_\mu \phi \nabla_\nu \phi + \left(-\frac{1}{2} g_{\mu\nu} \right) \left(-\frac{1}{2} g^{\rho\sigma} \nabla_\rho \phi \nabla_\sigma \phi - V(\phi) \right) \right]. \tag{4.78}$$

We therefore have

$$\begin{aligned} T^{(\phi)}_{\mu\nu} &= -2 \frac{1}{\sqrt{-g}} \frac{\delta S_\phi}{\delta g^{\mu\nu}} \\ &= \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} g^{\rho\sigma} \nabla_\rho \phi \nabla_\sigma \phi - g_{\mu\nu} V(\phi). \end{aligned} \tag{4.79}$$

In flat spacetime this reduces to what we had asserted, in Chapter 1, was the correct energy-momentum tensor for a scalar field.

On the other hand, in Minkowski space there is an alternative definition for the energy-momentum tensor, which is sometimes given in books on electromagnetism or field theory. In this context energy-momentum conservation arises

as a consequence of symmetry of the Lagrangian under spacetime translations. **Noether's theorem** states that every symmetry of a Lagrangian implies the existence of a conservation law; invariance under the four spacetime translations leads to a tensor $S^{\mu\nu}$, which obeys $\partial_\mu S^{\mu\nu} = 0$ (four relations, one for each value of ν). The details can be found in Wald (1984) or Peskin and Schroeder (1995). Applying Noether's procedure to a Lagrangian that depends on some fields Φ^i and their first derivatives $\partial_\mu \Phi^i$ (in flat spacetime), we obtain

$$S^{\mu\nu} = \frac{\delta \mathcal{L}}{\delta(\partial_\mu \Phi^i)} \partial^\nu \Phi^i - \eta^{\mu\nu} \mathcal{L}, \tag{4.80}$$

where a sum over i is implied. You can check that this tensor is conserved by virtue of the equations of motion of the matter fields. $S^{\mu\nu}$ often goes by the name "canonical energy-momentum tensor"; however, there are a number of reasons why it is more convenient for us to use (4.75). First, (4.75) is in fact what appears on the right hand side of Einstein's equation when it is derived from an action, and it is not always possible to generalize (4.80) to curved spacetime. But even in flat space (4.75) has its advantages; it is manifestly symmetric, and also guaranteed to be gauge invariant, neither of which is true for (4.80). We will therefore stick with (4.75) as the definition of the energy-momentum tensor.

Now that Einstein's equation has been derived, the rest of this chapter is devoted to exploring some of its properties. These discussions are fascinating but not strictly necessary; if you like, you can jump right to the applications discussed in subsequent chapters.

4.4 ■ PROPERTIES OF EINSTEIN'S EQUATION

Einstein's equation may be thought of as a set of second-order differential equations for the metric tensor field $g_{\mu\nu}$. There are really ten independent equations (since both sides are symmetric two-index tensors), which seems to be exactly right for the ten unknown functions of the metric components. However, the Bianchi identity $\nabla^\mu G_{\mu\nu} = 0$ represents four constraints on the functions $R_{\mu\nu}(x)$, so there are only six truly independent equations in (4.44). In fact this is appropriate, since if a metric is a solution to Einstein's equation in one coordinate system x^μ it should also be a solution in any other coordinate system $x^{\mu'}$. This means that there are four unphysical degrees of freedom in $g_{\mu\nu}$, represented by the four functions $x^{\mu'}(x^\mu)$, and we should expect that Einstein's equation only constrains the six coordinate-independent degrees of freedom.

As differential equations, these are extremely complicated; the Ricci scalar and tensor are contractions of the Riemann tensor, which involves derivatives and products of the Christoffel symbols, which in turn involve the inverse metric and derivatives of the metric. Furthermore, the energy-momentum tensor $T_{\mu\nu}$ will generally involve the metric as well. The equations are also nonlinear, so that two known solutions cannot be superposed to find a third. It is therefore very

difficult to solve Einstein's equation in any sort of generality, and it is usually necessary to make some simplifying assumptions. Even in vacuum, where we set the energy-momentum tensor to zero, the resulting equation (4.46) can be very difficult to solve. The most popular sort of simplifying assumption is that the metric has a significant degree of symmetry, and we will see later how isometries make life easier.

The nonlinearity of general relativity is worth a remark. In Newtonian gravity the potential due to two point masses is simply the sum of the potentials for each mass, but clearly this does not carry over to general relativity outside the weak-field limit. There is a physical reason for this, namely that in GR the gravitational field couples to itself. This can be thought of as a consequence of the equivalence principle—if gravitation did not couple to itself, a gravitational atom (two particles bound by their mutual gravitational attraction) would have a different inertial mass than gravitational mass (due to the negative binding energy). The nonlinearity of Einstein's equation is a reflection of the back-reaction of gravity on itself.

A nice way to think about this is provided by Feynman diagrams. These are used in quantum field theory to calculate the amplitudes for scattering processes, which can be obtained by summing the various contributions from different interactions, each represented by its own diagram. Even if we don't go so far as to quantize gravity and calculate scattering cross-sections (see the end of this section), we can still draw Feynman diagrams as a simple way of keeping track of which interactions exist and which do not. A simple example is provided by the electromagnetic interaction between two electrons; this can be thought of as due to exchange of a virtual photon, as shown in Figure 4.1.

In contrast, there is no diagram in which two photons exchange another photon between themselves, because electromagnetism is linear (there is no back-reaction). The gravitational interaction, meanwhile, can be thought of as deriving from the exchange of a virtual graviton (a quantized perturbation of the metric). The nonlinearity manifests itself as the fact that both electrons and gravi-

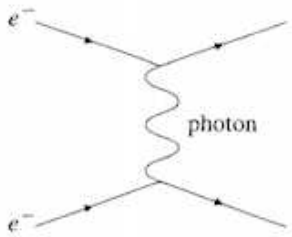


FIGURE 4.1 A Feynman diagram for electromagnetism. In quantum field theory, such diagrams are used to calculate amplitudes for scattering processes; here, just think of it as a cartoon representing a certain interaction. The point of this particular diagram is that the coupling of photons to electrons is what causes the electromagnetic interaction between them. In contrast, there is no coupling of photons to other photons, and no analogous diagram in which photons interact.

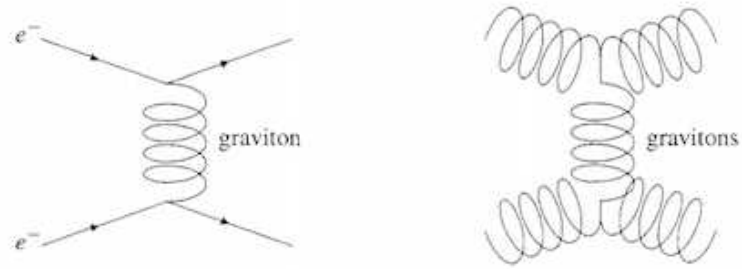


FIGURE 4.2 Feynman diagrams for gravity. Upon quantization, Einstein's equation predicts spin-two particles called gravitons. We don't know how to carry out such a quantization consistently, but the existence of gravitons is sufficiently robust that it is expected to be a feature of any well-defined scheme. Since gravity couples to energy-momentum, gravitons interact with every kind of particle, including other gravitons. This provides a way of thinking about the nonlinearity of Einstein's theory.

tons can exchange virtual gravitons, and therefore exert a gravitational force, as shown in Figure 4.2. There is nothing unique about this feature of gravity; it is shared by most gauge theories, such as quantum chromodynamics, the theory of the strong interactions. Electromagnetism is actually the exception; the linearity can be traced to the fact that the relevant gauge group, U(1), is abelian. But nonlinearity does represent a departure from the Newtonian theory. This difference is experimentally detectable; the reason why (as we shall see) the orbit of Mercury is different in GR versus Newtonian gravity is that the gravitational field influences itself, and the closer we get to the Sun, the more noticeable that influence is.

Beyond the fact that it is complicated and nonlinear, it is worth thinking a bit about what Einstein's equation is actually telling us. Clearly it relates the energy-momentum distribution to components of the curvature tensor; but from a physical point of view, precisely what kind of gravitational field is generated by a given kind of source? One way to answer this question is to consider the evolution of the *expansion* θ of a family of neighboring timelike geodesics. We imagine a small ball of free test particles moving along geodesics with four-velocities U^μ, and follow their evolution; the expansion $\theta = \nabla_\mu U^\mu$ tells us how the volume of the ball is growing (or shrinking, if $\theta < 0$) at any one moment of time. Clearly the value of the expansion will depend on the initial conditions for our test particles. The effects of gravity, on the other hand, are encoded in the *evolution* of the expansion, which is governed by Raychaudhuri's equation. This equation, discussed in Appendix F, tells us that the derivative of the expansion with respect to the proper time τ along the geodesics is given by the following expression:

$$\frac{d\theta}{d\tau} = 2\omega^2 - 2\sigma^2 - \frac{1}{3}\theta^2 - R_{\mu\nu}U^\mu U^\nu. \tag{4.81}$$

The terms on the right-hand side are explained carefully in Appendix F; ω encodes the rotation of the geodesics, σ encodes the shear, and $R_{\mu\nu}$ is of course the Ricci tensor. Raychaudhuri's equation is a purely geometric relation, making no

reference to Einstein's equation. The combination of the two equations, however, can be used to describe how energy-momentum influences the motion of test particles, since Einstein's equation relates $T_{\mu\nu}$ to $R_{\mu\nu}$ and Raychaudhuri's equation relates $R_{\mu\nu}$ to $d\theta/d\tau$.

Let us consider the simplest possible situation, where we start with all of the nearby particles at rest with respect to each other in a small region of spacetime. Then the expansion, rotation, and shear will all vanish at this initial moment. Let us further construct locally inertial coordinates $x^{\hat{\mu}}$, in which $U^{\hat{\mu}}$ is in its rest frame, so that $U^{\hat{\mu}} = (1, 0, 0, 0)$ and $R_{\hat{\mu}\hat{\nu}}U^{\hat{\mu}}U^{\hat{\nu}} = R_{\hat{0}\hat{0}}$. We therefore have (in these coordinates, at this point)

$$\frac{d\theta}{d\tau} = -R_{\hat{0}\hat{0}}. \tag{4.82}$$

Now we can turn to Einstein's equation, in the form

$$R_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \tfrac{1}{2}Tg_{\mu\nu}\right). \tag{4.83}$$

Since we are in locally inertial coordinates, we have

$$g_{\hat{\mu}\hat{\nu}} = \eta_{\hat{\mu}\hat{\nu}} \tag{4.84}$$

$$T = g^{\hat{\mu}\hat{\nu}}T_{\hat{\mu}\hat{\nu}} = -\rho + p_x + p_y + p_z, \tag{4.85}$$

where $\rho = T_{\hat{0}\hat{0}}$ is the rest-frame energy density and $p_k = T_{\hat{k}\hat{k}}$ is the pressure in the $x^{\hat{k}}$ direction. Thus, (4.82) becomes

$$\frac{d\theta}{d\tau} = -4\pi G(\rho + p_x + p_y + p_z). \tag{4.86}$$

This equation is telling us that energy and pressure create a gravitational field that works to decrease the volume of our initially stationary ball of test particles (if ρ and the p_i's are all positive). In other words, gravity is attractive.

Of course, from (4.86) we see that gravity is not *necessarily* attractive; we could imagine sources for which $\rho+p_x+p_y+p_z$ were a negative number. Clearly, the role of pressure bears noting. For one thing, it represents an unambiguous departure from Newtonian theory, in which the pressure does not influence gravity (it doesn't appear in Poisson's equation, $\nabla^2\Phi = 4\pi G\rho$). The difference is hard to notice in our Solar System, since the pressure in the Sun and planets is much less than the energy density, which is dominated by the rest masses of the constituent particles. For another thing, notice that the *gravitational* effect of the pressure is opposite to that of the *direct* effect with which we are more familiar, namely that positive pressure works to push things apart. In most circumstances the direct effect of pressure is much more noticeable. However, the pressure can only act directly when there is a pressure gradient (for example, a change in pressure between the interior and exterior of a piston), whereas the gravitational effect depends only on the value of the pressure locally. If there were a perfectly smooth

pressure, it would only be detectable through its gravitational effect; an example is provided by vacuum energy, discussed in Section 4.5.

As a final comment on (4.86), let's point out that it is completely equivalent to Einstein's equation—they convey identical information. This very specific relation will hold for any set of initially motionless test particles; the only way this can happen is if all of the components of Einstein's equation are true. If we like, then, we can state Einstein's equation in words[1] as follows: "The expansion of the volume of any set of particles initially at rest is proportional to (minus) the sum of the energy density and the three components of pressure."

So Einstein's equation tells us that energy density and pressure affect the Ricci tensor in such a way as to attract particles together when ρ and p are positive. What about the components of the Riemann tensor that are not included in the Ricci tensor? In Chapter 3 we found that these components were described by the Weyl tensor (expressed here in four dimensions),

$$C_{\rho\sigma\mu\nu} = R_{\rho\sigma\mu\nu} + \tfrac{1}{3} g_{\rho[\mu} g_{\nu]\sigma} R - g_{\rho[\mu} R_{\nu]\sigma} + g_{\sigma[\mu} R_{\nu]\rho}. \tag{4.87}$$

The Ricci tensor is the trace of the Riemann tensor, while the Weyl tensor describes the trace-free part; together they provide a complete characterization of the curvature. Clearly, given some specified energy-momentum distribution, there is still some freedom in the choice of Weyl curvature, since there is no analogue of Einstein's equation to relate $C^{\rho}{}_{\sigma\mu\nu}$ algebraically to $T_{\mu\nu}$. This is exactly as it should be. Imagine for example a spacetime that is vacuum everywhere, $R_{\mu\nu} = 0$. Flat Minkowski space is a possible solution in such a case, but so is a gravitational wave propagating through empty spacetime (as we will discuss in Chapter 7).

Since only $R_{\mu\nu}$ enters Einstein's equation, it might appear that the components of $C_{\rho\sigma\mu\nu}$ are completely unconstrained. But recall that we are not permitted to arbitrarily specify the components of the curvature tensor throughout a manifold; they are related by the Bianchi identity,

$$\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0. \tag{4.88}$$

As you showed in Exercise 10 of Chapter 3, this identity implies a differential relation for the Weyl tensor of the form

$$\nabla^{\rho} C_{\rho\sigma\mu\nu} = \nabla_{[\mu} R_{\nu]\sigma} + \tfrac{1}{6} g_{\sigma[\mu} \nabla_{\nu]} R. \tag{4.89}$$

On the right-hand side, the Riemann tensor only appears via its contractions the Ricci scalar and tensor, which can be related to $T_{\mu\nu}$ by Einstein's equation; we therefore have

$$\nabla^{\rho} C_{\rho\sigma\mu\nu} = 8\pi G \left(\nabla_{[\mu} T_{\nu]\sigma} + \tfrac{1}{3} g_{\sigma[\mu} \nabla_{\nu]} T \right). \tag{4.90}$$

So, while $R_{\mu\nu}$ and $T_{\mu\nu}$ are related algebraically through Einstein's equation, $C_{\rho\sigma\mu\nu}$ and $T_{\mu\nu}$ are related by this first-order differential equation. There will be

[1] J.C Baez, "The Meaning of Einstein's Equation," http://arXiv.org/abs/gr-qc/0103044.

a number of possible solutions for a given energy-momentum distribution, each specified by certain boundary conditions. This equation can be thought of as a propagation equation for gravitational waves, in close analogy with Maxwell's equations $\nabla_\mu F^{\nu\mu} = J^\nu$.

Having listed all of these lovely properties of Einstein's equation, it seems only fair that we should mention one distressing feature: the well-known difficulty of reconciling general relativity with quantum mechanics. GR is a classical field theory: the dynamical variable is a field (the metric) defined on spacetime, and coordinate-invariant quantities constructed from this field (such as the curvature scalar) can in principle be specified and measured to arbitrary accuracy. In the case of other field theories, such as electromagnetism, there are well-understood procedures for beginning with the classical theory and quantizing it, to obtain the dynamics of operators acting on wave functions living in a Hilbert space. For GR, the usual procedures run into both technical and conceptual difficulties, a description of which is beyond the scope of this book. One aspect of the technical difficulties is that GR is not "renormalizable" in the way that the Standard Model of particle physics is; when considering higher-order quantum effects, infinities appear that cannot be absorbed in any finite number of parameters. Nonrenormalizability does not mean that theory is fundamentally incorrect, but is a strong suggestion that it should only be taken seriously up to a certain energy scale.

Fortunately, the regime in which observable effects of quantum gravity are expected to become important is far from our everyday experience (or, for that matter, any conditions we can produce in the lab). Way back in 1899 Planck noticed that his constant h, for which nowadays we more often substitute $\hbar = h/2\pi = 1.05 \times 10^{-27}$ cm^2 g/sec, could be combined with Newton's constant $G = 6.67 \times 10^{-8}$cm^3 g^{-1} sec^{-2} and the speed of light $c = 3.00 \times 10^{10}$cm sec^{-1} to form a basic set of dimensionful quantities: the Planck mass,

$$m_P = \left(\frac{\hbar c}{G}\right)^{1/2} = 2.18 \times 10^{-5}\,\text{g}, \tag{4.91}$$

the Planck length,

$$l_P = \left(\frac{\hbar G}{c^3}\right)^{1/2} = 1.62 \times 10^{-33}\,\text{cm}, \tag{4.92}$$

the Planck time,

$$t_P = \left(\frac{\hbar G}{c^5}\right)^{1/2} = 5.39 \times 10^{-44}\,\text{sec}, \tag{4.93}$$

and the Planck energy,

$$E_P = \left(\frac{\hbar c^5}{G}\right)^{1/2} = 1.95 \times 10^{16}\,\text{erg} \tag{4.94}$$

$$= 1.22 \times 10^{19}\,\text{GeV}. \tag{4.95}$$

A GeV is 10^9 electron volts, a common unit in particle physics, as it is approximately the mass of a proton. We usually set $\hbar = c = 1$, so that these quantities are all indistinguishable in the sense that $m_P = l_P^{-1} = t_P^{-1} = E_P$. You will hear people say things like "the Planck mass is 10^{19} GeV"; or simply refer to "the Planck scale." Another commonly used quantity is the reduced Planck scale, $\bar{m}_P = m_P/\sqrt{8\pi} = 2.43 \times 10^{18}$ GeV, which is often more convenient in equations—note that the coefficient of the curvature scalar in (4.73) is $\bar{m}_P^2/2$. Most likely, quantum gravity does not become important until we consider particle masses greater than m_P, or times shorter than t_P, or lengths smaller than l_P, or energies higher than E_P; at lower scales, classical GR should suffice. Since these are all far removed from observable phenomena, constructing a consistent theory of quantum gravity is more an issue of principle than of practice. On the other hand, quantum effects in curved spacetime might be important in the real world; as we will discuss in Chapter 8, they might lead to density fluctuations in the early universe, which grow into the galaxies and large-scale structure we observe today.

There is a leading contender for a fully quantum theory that would encompass GR in the appropriate limit: string theory. In string theory we imagine that the fundamental objects are not point particles like electrons or photons, but rather small one-dimensional objects called strings, which can be either closed loops or open segments. String theory was originally proposed as a model of the strong nuclear force, but it was soon realized that the theory inevitably predicted a massless spin-two particle: exactly what a quantum theory of gravity would require. String theory seems to be a consistent quantum theory, and it predicts gravity, but there is still a great deal about it that we don't understand. In particular, the way in which a classical spacetime arises out of fundamental strings is somewhat mysterious, and the connection to direct experiments is tenuous at best. Nevertheless, string theory is remarkably rich and robust, and promises to be an important part of theoretical physics for the foreseeable future.

4.5 ■ THE COSMOLOGICAL CONSTANT

A characteristic feature of general relativity is that the source for the gravitational field is the entire energy-momentum tensor. In nongravitational physics, only *changes* in energy from one state to another are measurable; the normalization of the energy is arbitrary. For example, the motion of a particle with potential energy $V(x)$ is precisely the same as that with a potential energy $V(x) + V_0$, for any constant V_0. In gravitation, however, the actual value of the energy matters, not just the differences between states.

This behavior opens up the possibility of **vacuum energy**: an energy density characteristic of empty space. One feature that we might want the vacuum to exhibit is that it not pick out a preferred direction; it will still be possible to have a nonzero energy density if the associated energy-momentum tensor is Lorentz invariant in locally inertial coordinates. Lorentz invariance implies that the corre-

sponding energy-momentum tensor should be proportional to the metric,

$$T^{(\mathrm{vac})}_{\hat{\mu}\hat{\nu}} = -\rho_{\mathrm{vac}}\eta_{\hat{\mu}\hat{\nu}}, \tag{4.96}$$

since $\eta_{\hat{\mu}\hat{\nu}}$ is the only Lorentz invariant (0, 2) tensor. This generalizes straightforwardly from inertial coordinates to arbitrary coordinates as

$$T^{(\mathrm{vac})}_{\mu\nu} = -\rho_{\mathrm{vac}}g_{\mu\nu}. \tag{4.97}$$

Comparing to the perfect-fluid energy-momentum tensor $T_{\mu\nu} = (\rho + p)U_\mu U_\nu + pg_{\mu\nu}$, we find that the vacuum looks like a perfect fluid with an isotropic pressure opposite in sign to the energy density,

$$p_{\mathrm{vac}} = -\rho_{\mathrm{vac}}. \tag{4.98}$$

The energy density should be constant throughout spacetime, since a gradient would not be Lorentz invariant.

If we decompose the energy-momentum tensor into a matter piece $T^{(\mathrm{M})}_{\mu\nu}$ and a vacuum piece $T^{(\mathrm{vac})}_{\mu\nu} = -\rho_{\mathrm{vac}}g_{\mu\nu}$, Einstein's equation is

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} = 8\pi G\left(T^{(\mathrm{M})}_{\mu\nu} - \rho_{\mathrm{vac}}g_{\mu\nu}\right). \tag{4.99}$$

Soon after inventing GR, Einstein tried to find a static cosmological model, since that was what astronomical observations of the time seemed to imply. The result was the Einstein static universe, which will be discussed in Chapter 8. In order for this static cosmology to solve the field equation with an ordinary matter source, it was necessary to add a new term called the **cosmological constant**, Λ, which enters as

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{4.100}$$

From comparison with (4.99), we see that the cosmological constant is precisely equivalent to introducing a vacuum energy density

$$\rho_{\mathrm{vac}} = \frac{\Lambda}{8\pi G}. \tag{4.101}$$

The terms "cosmological constant" and "vacuum energy" are essentially interchangeable.

Is a nonzero vacuum energy something we should expect? We arrived at the Hilbert Lagrangian $\widehat{\mathcal{L}}_H = R$ by looking for the simplest possible scalar we could construct from the metric. Of course there is an even simpler one, namely a constant. Using (4.69), it is straightforward to check that

$$S = \int d^4x\,\sqrt{-g}\left[\frac{1}{16\pi G}(R - 2\Lambda) + \widehat{\mathcal{L}}_{\mathrm{M}}\right] \tag{4.102}$$

leads to the modified equation (4.100); alternatively, the vacuum Lagrangian is simply

$$\widehat{\mathcal{L}}_{\text{vac}} = -\rho_{\text{vac}}. \tag{4.103}$$

So it is certainly easy to introduce vacuum energy; however, we have no insight into its expected value, since it enters as an arbitrary constant.

The vacuum energy ultimately is a constant of nature in its own right. (An exception occurs in certain theories where a spacetime symmetry such as supersymmetry or conformal invariance governs the value of the vacuum energy; here we are considering a more generic field theory.) Nevertheless, there are various distinct contributions to the vacuum energy, and it would be strange if the total value were much smaller than the individual contributions. One such contribution comes from zero-point fluctuations—the energies of quantum fields in their vacuum state.

Consider a simple harmonic oscillator, a particle moving in a one-dimensional potential $V(x) = \frac{1}{2}\omega^2 x^2$. Classically, the vacuum for this system is the state in which the particle is motionless and at the minimum of the potential ($x = 0$), for which the energy in this case vanishes. Quantum-mechanically, however, the uncertainty principle forbids us from isolating the particle both in position and momentum, and we find that the lowest energy state has an energy $E_0 = \frac{1}{2}\hbar\omega$ (where we have temporarily reintroduced explicit factors of $\hbar$ for clarity). Of course, in the absence of gravity, either system actually has a vacuum energy that is completely arbitrary; we could add any constant to the potential without changing the theory. But quantum fluctuations have changed the zero-point energy from our classical expectation.

A precisely analogous situation holds in field theory. If we take the Fourier transform of a free quantum field (one where we ignore interactions for simplicity), we find that it becomes an infinite number of harmonic oscillators in momentum space, as we discuss in Chapter 9. The frequency ω of each oscillator is $\omega = \sqrt{m^2 + k^2}$, where m is the mass of the field and k is the magnitude of the wave vector of the mode. If we set the classical vacuum energy to zero, each of these modes contributes a quantum zero-point energy of $\hbar\omega/2$. Formally, adding all of these contributions together yields an infinite result. If, however, we discard the very high-momentum modes on the grounds that we trust our theory only up to a certain ultraviolet momentum cutoff k_{max}, we find that the resulting energy density is of the form

$$\rho_{\text{vac}} \sim \hbar k_{\text{max}}^4. \tag{4.104}$$

This answer could have been guessed by dimensional analysis; the numerical constants that have been neglected will depend on the precise theory under consideration. If we are confident that we can use ordinary quantum field theory all the way up to the reduced Planck scale $\bar{m}_{\text{P}} = (8\pi G)^{-1/2} \sim 10^{18}$ GeV, we expect a contribution of order

$$\rho_{\text{vac}} \sim (10^{18}\ \text{GeV})^4 \sim 10^{112}\ \text{erg/cm}^3. \tag{4.105}$$

Field theory may fail earlier, although quantum gravity is the best reason we have to believe it will fail at any specific scale.

As we will discuss in Chapter 8, cosmological observations imply

$$|\rho_\Lambda^{(\text{obs})}| \leq (10^{-12}\ \text{GeV})^4 \sim 10^{-8}\ \text{erg/cm}^3, \tag{4.106}$$

much smaller than the naive expectation just derived. The ratio of (4.105) to (4.106) is the origin of the famous discrepancy of 120 orders of magnitude between the theoretical and observational values of the cosmological constant. We are free to imagine that the bare vacuum energy is adjusted so that the net cosmological constant is consistent with the limit (4.106), except for one problem: we know of no special symmetry that could enforce a vanishing vacuum energy while remaining consistent with the known laws of physics; this conundrum is the "cosmological constant problem." We will discuss the cosmological effects of vacuum energy more in Chapter 8.[2]

4.6 ■ ENERGY CONDITIONS

Sometimes it is useful to think about Einstein's equation without specifying the theory of matter from which $T_{\mu\nu}$ is derived. This leaves us with a great deal of arbitrariness; consider for example the question, What metrics obey Einstein's equation? In the absence of some constraints on $T_{\mu\nu}$, the answer is any metric at all; simply take the metric of your choice, compute the Einstein tensor $G_{\mu\nu}$ for this metric, and then demand that $T_{\mu\nu}$ be equal to $G_{\mu\nu}$. It will automatically be conserved, by the Bianchi identity. Our real concern is with the existence of solutions to Einstein's equation in the presence of "realistic" sources of energy and momentum, whatever that means. One strategy is to consider specific kinds of sources, such as scalar fields, dust, or electromagnetic fields. However, we occasionally wish to understand properties of Einstein's equations that hold for a variety of different sources. In this circumstance it is convenient to impose *energy conditions* that limit the arbitrariness of $T_{\mu\nu}$.

Energy conditions are coordinate-invariant restrictions on the energy-momentum tensor. We must therefore construct scalars from $T_{\mu\nu}$, which is typically accomplished by contracting with arbitrary timelike vectors t^μ or null vectors ℓ^μ. For example, the weak energy condition (WEC) states that $T_{\mu\nu}t^\mu t^\nu \geq 0$ for all timelike vectors t^μ. For purposes of physical intuition, it is useful to consider the special case where the source is a perfect fluid, so that the energy-momentum tensor takes the form

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu}, \tag{4.107}$$

where U^μ is the fluid four-velocity. Let's use this form to translate the WEC into physical terms. Because the pressure is isotropic, $T_{\mu\nu}t^\mu t^\nu$ will be nonnegative

[2]For more on the physics and cosmology of vacuum energy, see S.M. Carroll, *Liv. Rev. Rel.* **4**, 1 (2001), `http://arxiv.org/astro-ph/0004075`.

for all timelike vectors t^μ if both $T_{\mu\nu}U^\mu U^\nu \geq 0$ and $T_{\mu\nu}\ell^\mu \ell^\nu \geq 0$ for some null vector ℓ^μ (convince yourself of this; it's just adding vectors). We therefore evaluate

$$T_{\mu\nu}U^\mu U^\nu = \rho, \quad T_{\mu\nu}\ell^\mu \ell^\nu = (\rho + p)(U_\mu \ell^\mu)^2. \tag{4.108}$$

The WEC therefore implies $\rho \geq 0$ and $\rho + p \geq 0$. These are simply the reasonable-sounding requirements that the energy density be nonnegative and the pressure not be too large compared to the energy density. Of course we need not restrict ourselves to perfect fluids, we merely use them to gain insight into the requirements the energy conditions impose.

There are a number of different energy conditions, appropriate to different circumstances. Some of the most popular are the following:

- The **Weak Energy Condition** or WEC, as just discussed, states that $T_{\mu\nu}t^\mu t^\nu \geq 0$ for all timelike vectors t^μ, or equivalently that $\rho \geq 0$ and $\rho + p \geq 0$.

- The **Null Energy Condition** or NEC states that $T_{\mu\nu}\ell^\mu \ell^\nu \geq 0$ for all null vectors ℓ^μ, or equivalently that $\rho + p \geq 0$. It is a special case of the WEC, with the timelike vector replaced by a null vector. The energy density may now be negative, so long as there is a compensating positive pressure.

- The **Dominant Energy Condition** or DEC includes the WEC ($T_{\mu\nu}t^\mu t^\nu \geq 0$ for all timelike vectors t^μ), as well as the additional requirement that $T^{\mu\nu}t_\mu$ is a nonspacelike vector (namely, that $T_{\mu\nu}T^\nu{}_\lambda t^\mu t^\lambda \leq 0$). For a perfect fluid, these conditions together are equivalent to the simple requirement that $\rho \geq |p|$; the energy density must be nonnegative, and greater than or equal the magnitude of the pressure.

- The **Null Dominant Energy Condition** or NDEC is the DEC condition for null vectors only: for any null vector ℓ^μ, $T_{\mu\nu}\ell^\mu \ell^\nu \geq 0$ and $T^{\mu\nu}\ell_\mu$ is a nonspacelike vector. The allowed density and pressure are the same as for the DEC, except that negative densities are allowed so long as $p = -\rho$. In other words, the NDEC excludes all sources excluded by the DEC, except for a negative vacuum energy.

- The **Strong Energy Condition** or SEC states that $T_{\mu\nu}t^\mu t^\nu \geq \frac{1}{2}T^\lambda{}_\lambda t^\sigma t_\sigma$ for all timelike vectors t^μ, or equivalently that $\rho + p \geq 0$ and $\rho + 3p \geq 0$. Note that the SEC does *not* imply the WEC. It implies the NEC, along with excluding excessively large negative pressures. From (4.86) we see that it is the SEC that implies gravitation is attractive.

These conditions are illustrated in Figure 4.3. In addition we have plotted the constraint $w \geq -1$, where $w = p/\rho$ is called the **equation-of-state parameter**. This is a useful concept in cosmology, where sources often have equations of state $p = w\rho$ with w being a constant (of course, w is defined whether it is constant

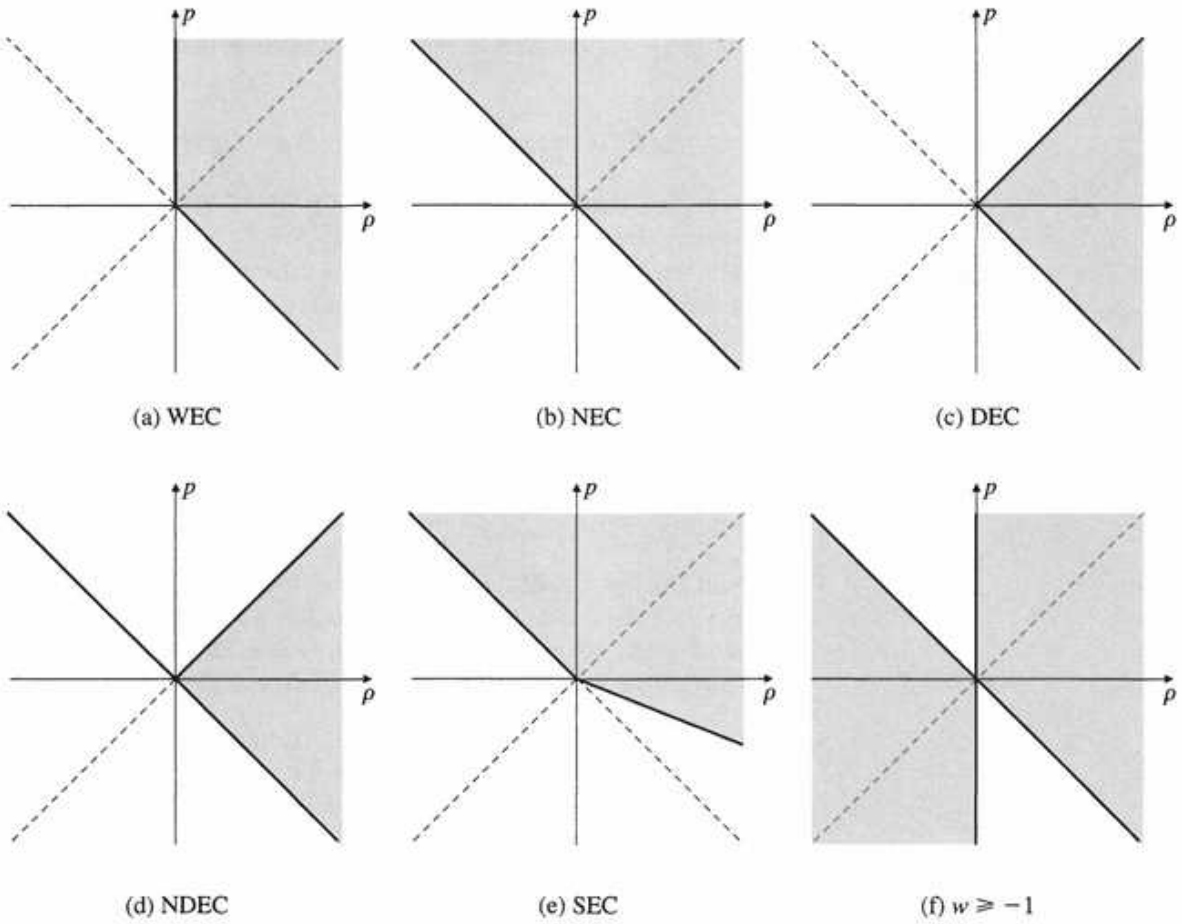


FIGURE 4.3 Energy conditions as applied to perfect fluids, expressed as allowed regions of energy density ρ and pressure p. Illustrated are the Weak Energy Condition (WEC), Null Energy Condition (NEC), Dominant Energy Condition (DEC), Null Dominant Energy Condition (NDEC), and the Strong Energy Condition (SEC). For comparison, we also have illustrated the condition $w \geq -1$, where $w = p/\rho$ is the equation-of-state parameter.

or not). If we restrict ourselves to sources with $\rho \geq 0$, then any of the energy conditions mentioned above will imply $w \geq -1$.

Most ordinary classical forms of matter, including scalar fields and electromagnetic fields, obey the DEC (see Exercises), and hence the less restrictive conditions (WEC, NEC, NDEC). The SEC is useful in the proof of some singularity theorems, but can be violated by certain forms of matter, such as a massive scalar field. It turns out that quantum fields can generically violate any of the energy conditions we have listed; there may, however, be inequalities involving integrals over regions of spacetime that are satisfied even by quantum fields. This is an area of current investigation.

The energy conditions are not, strictly speaking, related to energy conservation; the Bianchi identity guarantees that $\nabla_\mu T^{\mu\nu} = 0$ regardless of whether we

impose any additional constraints on $T^{\mu\nu}$. Rather, they serve to prevent other properties that we think of as "unphysical," such as energy propagating faster than the speed of light, or empty space spontaneously decaying into compensating regions of positive and negative energy. In particular, Hawking and Ellis (1973) prove a *conservation theorem*: Essentially, if the energy-momentum tensor obeys the DEC and vanishes in some spacelike region, then it will necessarily vanish everywhere in the future domain of dependence of that region (see Section 2.7 for the definition of the future domain of dependence). Thus, energy cannot spontaneously appear from nothing, nor can it sneak outside the light cone. The theorem does not include the converse statement (that sources violating the DEC are necessarily acausal), so it pays to be careful.

4.7 ■ THE EQUIVALENCE PRINCIPLE REVISITED

In this section we will examine more carefully the underpinnings and consequences of the Principle of Equivalence, which we used in Section 4.1 to motivate the minimal-coupling procedure for generalizing physics to curved spacetime. We will see that the Principle of Equivalence is not a sacred physical law, nor is it even a mathematically rigorous statement; at a more fundamental level, it arises as a consequence of the nature of general relativity as an effective field theory valid at macroscopic distances, and our job is to determine which kinds of couplings between matter and the metric we would expect in such a theory.

In practice, it is common to invoke the Equivalence Principle to justify any of the following four ideas:

1. Laws of physics should be expressed (or at least be expressible) in generally covariant form.
2. There exists a metric on spacetime, the curvature of which we interpret as gravity.
3. There do not exist any other fields that resemble gravity.
4. The interactions of matter fields to curvature are minimal: they do not involve direct couplings to the Riemann tensor or its contractions.

These very different statements each have a very different status: the first is vacuous, the second is both profound and almost certainly true, the third is interesting and testable, and the fourth is just a useful approximation. Let's examine each of them in turn.

The first statement is sometimes called the Principle of Covariance. It is more or less content-free. "Generally covariant" simply means that all of the terms in an equation transform in the same way under a change of coordinates, so that the form of the equation is coordinate-invariant. Due to the universal nature of the tensor transformation law, the most straightforward way of achieving this aim is to make the equation manifestly tensorial. Certainly there is nothing wrong if a law is expressed in a form that is not generally covariant, as long as we

know that it is possible to rewrite it in a coordinate-independent way. On the other hand, it is *always* possible to write laws in a coordinate-independent way, if the laws are well-defined to begin with. A physical system acting in a certain way doesn't know which coordinate system you are using to describe it; consequently, anything deserving of the name "law of physics" (as opposed to some particular statement of that law) must be independent of coordinates. An insistence on explicit coordinate-independence says nothing about the adaptation of laws to curved spacetime; as we have seen, manifestly tensorial equations take on the same form regardless of the geometry.

Consider Maxwell's equations in flat spacetime, as we wrote them in Chapter 1:

$$\partial_\mu F^{\nu\mu} = J^\nu. \tag{4.109}$$

The right-hand side is a well-defined tensor, while the left-hand side is not, due to the appearance of the partial derivative. That's okay, since we know that this equation is valid only in inertial coordinates in Minkowski space. A coordinate-invariant way of expressing the same law is

$$\nabla_\mu F^{\nu\mu} = J^\nu. \tag{4.110}$$

No physical principle needs to be invoked to conclude that this is the correct formulation in Minkowski space; it is the *unique* tensorial equation, which is equivalent to (4.109) in inertial coordinates. It is not the unique generalization to curved spacetime, since we could imagine new terms involving products of $F_{\mu\nu}$ and $R^\rho{}_{\sigma\mu\nu}$; the status of such additional terms is directly addressed by the minimal-coupling assumption, point four in the above list. By itself, however, making things "tensorial" or "generally covariant" is a simple matter of logical necessity, not a physical principle that one could imagine disproving by experiment. (Another spin on the same idea is "diffeomorphism invariance," discussed in the Appendix B.)

The second purported consequence of the Equivalence Principle from our list above is much deeper, and by no means obvious. Although he was inspired by the EP, this geometric insight was Einstein's great breakthrough. At the beginning of Chapter 2 we discussed why such an insight was warranted: the EP implies that gravity is universal, which implies in turn that gravitational fields become impossible to measure in small regions of spacetime, a feature which in turn is most directly implemented by identifying gravitation with the effects of spacetime geometry. These steps are well-motivated suggestions, not rigorously derived consequences; once we have the idea that there is a metric whose curvature gives rise to gravity, we can check its usefulness by comparing with experiment. As we've mentioned, it passes with flying colors. An accumulation of evidence (such as the gravitational redshift discussed in Chapter 2) is consistent with the idea that idealized rods and clocks behave as they should if the geometry of spacetime were curved. Still, one should not imagine proving that there really is a metric with the desired properties; we make the hypothesis, test it against ever-more precise ex-

periments, and deduce its range of usefulness. Indeed, the demands of eventually reconciling general relativity with quantum mechanics suggest to many that the metric will ultimately be revealed as a concept derived from a more fundamental collection of degrees of freedom. For our present purposes this ultimate resolution doesn't matter; the idea of a curved metric has proven its usefulness beyond a reasonable doubt, and we work to extend our understanding of its properties until they run up against insurmountable obstacles (either theoretical or empirical).

Given our conviction that the effects of gravitation are best ascribed to the curvature of a metric on spacetime, what would we conclude if experiments were to detect an apparent violation of the Equivalence Principle? For example, we might imagine an experiment that revealed that the acceleration of test bodies in the direction of the Earth or Sun actually did depend, ever so slightly, on the composition of the test body. (The best current limits on such anomalous accelerations constrain them to be less than 10^{-12} times that due to gravity.)[3] In such a circumstance, nobody would really be tempted to declare that general relativity had been completely undermined and it was necessary to start over. Rather, we would return to the definition of "test body," which includes the proviso that the body be uncharged. An electron, for example, would not make a good test body, as it would be buffeted about by ambient electromagnetic fields as well as by gravity. Similarly, by far the most straightforward explanation of any hypothetical anomalous acceleration on purportedly neutral test bodies would be to imagine that we had discovered the existence of a new long-range field, under which our test bodies were actually charged. To have remain undetected thus far, such a field must be either very weakly coupled, or must couple almost universally, so as to mimic the effects of gravity. We could imagine, for example, scalar fields that couple to the trace of the energy-momentum tensor, or vector fields that couple to baryon number. The mass of ordinary test bodies is almost proportional to their baryon number, which counts the number of protons and neutrons in the body. It is therefore sometimes convenient to think of "tests of the Equivalence Principle" as tests of the third of our statements above—that there do not exist any other fields that resemble gravity (where a field resembles gravity if it is long-range and couples almost universally to mass). Again, detecting a violation of this hypothesis would be most directly interpreted as discovery of a new "fifth force" rather than as a repudiation of Einstein's ideas. As to whether we should expect to discover such a new field if we improve upon current experiments, it is hard to say; on the one hand, it is easy to concoct models with new long-range forces, but on the other hand, they would typically be strong enough to already have been detected. At this stage it is still worthwhile to keep an open mind.

Beyond the very existence of the metric, the heart of the Equivalence Principle lies in the fourth of our formulations, that the interactions of matter fields to curvature are minimal: they do not involve direct couplings to the Riemann tensor or its contractions. For example, we could consider the following possible alternative

[3] Y. Su et al., *Phys. Rev.* **D 50**, 3614 (1994).

to the conventional geodesic equation:

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = \alpha\,(\nabla_\sigma R)\frac{dx^\mu}{d\lambda}\frac{dx^\sigma}{d\lambda}, \tag{4.111}$$

where R is the Ricci scalar and α is a coupling constant. This equation also reduces to straight-line motion in flat spacetime, but would allow for direct detection of spacetime curvature in small regions by measurement of the coupling to $\nabla_\sigma R$. Why, then, does nature choose the simple geodesic equation? As a first step toward an answer, consider the dimensions of the coupling α. Since $c = 1$ and space and time have the same units, we can use length as our basic dimension. The metric, the inverse metric, and $dx^\mu/d\lambda$ are then dimensionless. The partial derivative operator has units of inverse length, as does the covariant derivative. The Christoffel symbols involve first derivatives of the metric, and thus have dimensions of inverse length; similarly, the Riemann tensor, Ricci tensor, and Ricci scalar have dimensions of inverse length squared:

$$\left[\frac{dx^\mu}{d\lambda}\right] = [g_{\mu\nu}] = [g^{\mu\nu}] = L^0, \quad [\nabla_\mu] = [\Gamma^\mu_{\rho\sigma}] = L^{-1}, \quad [R] = L^{-2}. \tag{4.112}$$

To be consistent, the coupling α must have dimensions of length squared:

$$[\alpha] = L^2. \tag{4.113}$$

The square root of α therefore defines a length scale; what should the length scale be? We don't know for sure, but there is every reason to believe it should be extremely small. There are two arguments for this. One is that, since the coupling represented by α is of gravitational origin, the only reasonable expectation for the relevant length scale is

$$\alpha \sim l_{\rm P}^2, \tag{4.114}$$

where $l_{\rm P}$ is the Planck length. Another reason is simply a more sophisticated version of this "what else could it be?" rationale. Although general relativity is a classical theory, at a deeper level we expect that it is merely an effective field theory describing an underlying quantum-mechanical structure. Even without knowing what this structure may be, a generic expectation (derived from our experience with quantum field theories we do understand) is that the effective classical limit should contain all possible interactions, but with dimensionful length parameters representing scales at which new degrees of freedom become important (recall our discussion of effective field theory at the end of Chapter 1). Thus, the Fermi theory of the weak interactions contains a length scale, which we now know to correspond to the scale of electroweak symmetry breaking where W and Z bosons become relevant. Since we do not expect new gravitational physics to arise before the Planck scale, the higher-order interactions associated with gravity should be suppressed by appropriate powers of the Planck length.

How much suppression does this represent? One measure would be to compare l_P (and thus the likely value of the parameter α) to a typical gravitational length scale near the vicinity of the Earth. The strength of gravity on Earth is characterized by the acceleration due to gravity, $a_g = 980\ \text{cm/sec}^2$. To construct a quantity with dimensions of length, we define

$$l_\oplus = c^2/a_g \sim 10^{18}\ \text{cm}, \tag{4.115}$$

where the symbol $\oplus$ in this context stands for the Earth (not a direct sum). So the relative strength of higher-order gravitational effects is measured by

$$\frac{l_P}{l_\oplus} \sim 10^{-51}. \tag{4.116}$$

In fact, since we expect $\alpha \sim l_P^2$, the suppression will be of order 10^{-102}. Consequently, there seems to be little need to worry about the possible role of such couplings. But dramatic departures should be kept in mind; recent ideas about large extra dimensions have opened up the possibility of observing direct gravitational interactions at particle accelerators. Ultimately, there is no way to resolve these problems by pure thought alone; only experiment can decide among the alternatives.

4.8 ■ ALTERNATIVE THEORIES

General relativity has passed a wide variety of experimental tests. Nevertheless, it is always possible that the next experiment we do will reveal a deviation from Einstein's original formulation. Let us therefore briefly consider ways in which general relativity could be modified. There are an uncountable number of such ways, but we will consider four different possibilities:

- gravitational scalar fields
- extra spatial dimensions
- higher-order terms in the action
- nonChristoffel connections

A popular set of alternative models are known as **scalar-tensor theories** of gravity, since they involve both the metric tensor, $g_{\mu\nu}$ and a scalar field, λ. In particular, the scalar field couples directly to the curvature scalar, not simply to the metric (as the Equivalence Principle would seem to imply). The action can be written as a sum of a gravitational piece, a pure-scalar piece, and a matter piece:

$$S = S_{fR} + S_\lambda + S_M, \tag{4.117}$$

where

$$S_{fR} = \int d^4x\sqrt{-g}\, f(\lambda)R, \tag{4.118}$$

$$S_\lambda = \int d^4x\sqrt{-g}\left[-\frac{1}{2}h(\lambda)g^{\mu\nu}(\partial_\mu\lambda)(\partial_\nu\lambda) - U(\lambda)\right], \tag{4.119}$$

and

$$S_{\rm M} = \int d^4x\sqrt{-g}\,\widehat{\mathcal{L}}_{\rm M}(g_{\mu\nu}, \psi_i). \tag{4.120}$$

Here, $f(\lambda)$, $h(\lambda)$ and $U(\lambda)$ are functions that define the theory, and the matter Lagrangian $\widehat{\mathcal{L}}_{\rm M}$ depends on the metric and a set of matter fields ψ_i, but not on λ. By change of variables we can always set $h(\lambda) = 1$, but we leave it here to facilitate comparison with models found in the literature.

The equations of motion for this theory include the gravitational equation (from varying with respect to the metric), and the scalar equation (from varying with respect to λ), as well as the appropriate matter equations. Let's start with the gravitational equation, which we can derive by following the same steps as for the ordinary Hilbert action (4.55). We consider perturbations of the metric,

$$g^{\mu\nu} \to g^{\mu\nu} + \delta g^{\mu\nu}. \tag{4.121}$$

Following the procedure from Section 4.3, the variation of the gravitational part of the action is

$$\delta S_{fR} = \int d^4x\sqrt{-g}\, f(\lambda)\Big[\Big(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\Big)\delta g^{\mu\nu} + \nabla_\sigma\nabla^\sigma(g_{\mu\nu}\delta g^{\mu\nu}) - \nabla_\mu\nabla_\nu(\delta g^{\mu\nu})\Big]. \tag{4.122}$$

For the Hilbert action, f is a constant, so the last two terms are total derivatives, which can be converted to surface terms through integration by parts and therefore ignored. Now integration by parts (twice) picks up derivatives of f, and we obtain

$$\delta S_{fR} = \int d^4x\sqrt{-g}\left[f(\lambda)G_{\mu\nu} + g_{\mu\nu}\Box f - \nabla_\mu\nabla_\nu f\right]\delta g^{\mu\nu}, \tag{4.123}$$

where $G_{\mu\nu}$ is the Einstein tensor. We have discarded surface terms as usual, although there are subtleties concerning boundary contributions in this case; see Wald (1984) for a discussion. The gravitational equation of motion, including contributions from S_λ and $S_{\rm M}$, is thus

$$G_{\mu\nu} = f^{-1}(\lambda)\left(\tfrac{1}{2}T^{(\rm M)}_{\mu\nu} + \tfrac{1}{2}T^{(\lambda)}_{\mu\nu} + \nabla_\mu\nabla_\nu f - g_{\mu\nu}\Box f\right), \tag{4.124}$$

where the energy-momentum tensors are $T^{(i)}_{\mu\nu} = -2(-g)^{-1/2}\delta S_i/\delta g^{\mu\nu}$; in particular,

$$T^{(\lambda)}_{\mu\nu} = h(\lambda)\nabla_\mu\lambda\nabla_\nu\lambda - g_{\mu\nu}\left[\tfrac{1}{2}h(\lambda)g^{\rho\sigma}\nabla_\rho\lambda\nabla_\sigma\lambda + U(\lambda)\right]. \tag{4.125}$$

From looking at the coefficient of $T^{(\mathrm{M})}_{\mu\nu}$ in (4.124), we see that when the scalar field is *constant* (or practically so), we may identify $f(\lambda) = 1/(16\pi G)$, as makes sense from the original action (4.118). Meanwhile, if λ varies slightly from point to point in spacetime, it would be interpreted as a spacetime-dependent Newton's constant. The dynamics that control this variation are determined by the equation of motion for λ, which is straightforward to derive as

$$h\,\Box\lambda + \tfrac{1}{2}h'g^{\mu\nu}\nabla_\mu\lambda\nabla_\nu\lambda - U' + f'R = 0, \tag{4.126}$$

where primes denote differentiation with respect to λ. Notice that if we set $h(\lambda) = 1$ to get a conventional kinetic term for the scalar, λ obeys a conventional scalar-field equation of motion, with an additional coupling to the curvature scalar. In the real world, we don't want $f(\lambda)$ to vary too much, as it would have observable consequences in the classic experimental tests of GR in the solar system, and also in cosmological tests such as primordial nucleosynthesis. This can be ensured either by choosing $U(\lambda)$ so that there is a minimum to the potential and λ cannot deviate too far without a large input of energy—in other words, λ has a large mass—or by choosing $f(\lambda)$ and $h(\lambda)$ so that large changes in λ give rise to relatively small changes in the effective value of Newton's constant.

One of the earliest scalar-tensor models is known as Brans–Dicke theory, and corresponds in our notation to the choices

$$f(\lambda) = \frac{\lambda}{16\pi}, \quad h(\lambda) = \frac{\omega}{8\pi\lambda}, \quad U(\lambda) = 0. \tag{4.127}$$

where ω is a coupling constant. The scalar-tensor action takes the form

$$S_{\mathrm{BD}} = \int d^4x\sqrt{-g}\left[\frac{\lambda}{16\pi}R - \frac{\omega}{16\pi}g^{\mu\nu}\frac{(\partial_\mu\lambda)(\partial_\nu\lambda)}{\lambda}\right]. \tag{4.128}$$

In the Brans–Dicke theory, the scalar field is massless, but in the $\omega \to \infty$ limit the field becomes nondynamical and ordinary GR is recovered. Current bounds from Solar System tests imply $\omega > 500$, so if there is such a scalar field it must couple only weakly to the Ricci scalar.

A popular approach to dealing with scalar-tensor theories is to perform a conformal transformation to bring the theory in to a form that looks like conventional GR. We define a conformal metric

$$\tilde{g}_{\mu\nu} = 16\pi\widetilde{G}\,f(\lambda)g_{\mu\nu}, \tag{4.129}$$

where $\widetilde{G}$ will become Newton's constant in the conformal frame. Using formulae for conformal transformations from the Appendix G, the action S_{fR} from (4.118)

becomes

$$S_{fR} = \int d^4x\sqrt{-g}\, f(\lambda)R$$
$$= \int d^4x\sqrt{-\tilde{g}}\,(16\pi\widetilde{G})^{-1}\left[\widetilde{R} - \frac{3}{2}\tilde{g}^{\rho\sigma} f^{-2}\left(\frac{df}{d\lambda}\right)^2 (\widetilde{\nabla}_\rho\lambda)(\widetilde{\nabla}_\sigma\lambda)\right], \tag{4.130}$$

where as usual we have integrated by parts and discarded surface terms. In the conformal frame, therefore, the curvature scalar appears by itself, not multiplied by any function of λ. This frame is sometimes called the **Einstein frame**, since Einstein's equations for the conformal metric $\tilde{g}_{\mu\nu}$ take on their conventional form. The original frame with metric $g_{\mu\nu}$ is called the **Jordan frame**, or sometimes the **string frame**. (String theory typically predicts a scalar-tensor theory rather than ordinary GR, and the string worldsheet responds to the metric $g_{\mu\nu}$.)

Before going on with our analysis of the conformally-transformed theory, consider what happens if we choose

$$f(\lambda) = e^{\lambda/\sqrt{3}}, \quad h(\lambda) = 0, \quad U(\lambda) = 0, \tag{4.131}$$

a specific choice for $f(\lambda)$, but turning off the pure scalar terms in S_λ. Then we notice that the Einstein frame action (4.130) actually includes a conventional kinetic term for the scalar, even though it wasn't present in the Jordan frame action (4.118). Even without an explicit kinetic term for λ, the degrees of freedom of this theory include a propagating scalar as well as the metric. This should hopefully become more clear after we examine the degrees of freedom of the gravitational field in Chapter 7. There we will find that the metric $g_{\mu\nu}$ actually includes scalar (spin-0) and vector (spin-1) degrees of freedom as well as the expected tensor (spin-2) degrees of freedom; however, with the standard Hilbert action, these degrees of freedom are constrained rather than freely propagating. What we have just found is that multiplying R by a scalar in the action serves to bring the scalar degree of freedom to life, which is revealed explicitly in the Einstein frame.

If we do choose to include the pure-scalar action S_λ, we obtain

$$S_{fR} + S_\lambda = \int d^4x\sqrt{-\tilde{g}}\left[\frac{\widetilde{R}}{16\pi\widetilde{G}} - \frac{1}{2}K(\lambda)\tilde{g}^{\rho\sigma}(\widetilde{\nabla}_\rho\lambda)(\widetilde{\nabla}_\sigma\lambda) - \frac{U(\lambda)}{(16\pi\widetilde{G})^2 f^2(\lambda)}\right], \tag{4.132}$$

where

$$K(\lambda) = \frac{1}{16\pi\widetilde{G}f^2}\left[fh + 3(f')^2\right]. \tag{4.133}$$

We can make our action look utterly conventional by defining a new scalar field ϕ via

$$\phi = \int K^{1/2}\, d\lambda, \tag{4.134}$$

in terms of which the action becomes

$$S_{fR} + S_\lambda = \int d^4x \sqrt{-\tilde{g}} \left[\frac{\tilde{R}}{16\pi \tilde{G}} - \frac{1}{2}\tilde{g}^{\rho\sigma}(\tilde{\nabla}_\rho \phi)(\tilde{\nabla}_\sigma \phi) - V(\phi) \right], \tag{4.135}$$

where

$$V(\phi) = \frac{U(\lambda(\phi))}{(16\pi \tilde{G})^2 f^2(\lambda(\phi))}. \tag{4.136}$$

Amazingly, in the Einstein frame we have a completely ordinary theory of a scalar field in curved spacetime. So long as $f(\lambda)$ is well-behaved, the variables $(\tilde{g}_{\mu\nu}, \phi)$ can be used instead of $(g_{\mu\nu}, \lambda)$, in the sense that varying with respect to the new variables is equivalent to starting with the original equations of motion (4.124) and (4.126) and then doing the transformations (4.129) and (4.134).

Finally, we add in the matter action (4.120). Varying with respect to $\tilde{g}_{\mu\nu}$ will yield an energy-momentum tensor in the Einstein frame. In the original variables $(g_{\mu\nu}, \lambda)$, we knew that $S_{\rm M}$ was independent of λ, but now it will depend on both of the new variables $(\tilde{g}_{\mu\nu}, \phi)$; we can use the chain rule to characterize this dependence. Let us also assume that $S_{\rm M}$ depends on $g_{\mu\nu}$ only algebraically, not through derivatives. This will hold for ordinary scalar-field or gauge-field matter; things become more complicated for fermions, which we won't discuss here. We obtain

$$\begin{aligned}
\tilde{T}_{\mu\nu} &\equiv -2\frac{1}{\sqrt{-\tilde{g}}}\frac{\delta S_{\rm M}}{\delta \tilde{g}^{\mu\nu}} \\
&= -2\frac{1}{\sqrt{-\tilde{g}}}\frac{\partial g^{\alpha\beta}}{\partial \tilde{g}^{\mu\nu}}\frac{\delta S_{\rm M}}{\delta g^{\alpha\beta}} \\
&= -2(16\pi \tilde{G} f)^{-1}\frac{1}{\sqrt{-g}}\delta^\alpha_\mu \delta^\beta_\nu \frac{\delta S_{\rm M}}{\delta g^{\alpha\beta}} \\
&= (16\pi \tilde{G} f)^{-1} T_{\mu\nu}.
\end{aligned} \tag{4.137}$$

A similar trick works for the coupling of matter to ϕ, which comes from varying $S_{\rm M}$ with respect to ϕ, using $g^{\alpha\beta} = 16\pi \tilde{G} f \tilde{g}^{\alpha\beta}$:

$$\begin{aligned}
\frac{\delta S_{\rm M}}{\delta \phi} &= \frac{\partial g^{\alpha\beta}}{\partial \phi}\frac{\delta S_{\rm M}}{\delta g^{\alpha\beta}} \\
&= \left(16\pi \tilde{G}\frac{df}{d\phi}\tilde{g}^{\alpha\beta}\right)\left(-\frac{1}{2}\sqrt{-g}T^{\rm M}_{\alpha\beta}\right) \\
&= -\frac{1}{2f}\frac{df}{d\phi}\sqrt{-\tilde{g}}\tilde{T}^{\rm M},
\end{aligned} \tag{4.138}$$

where

$$\widetilde{T}^{(\mathrm{M})} = \tilde{g}^{\alpha\beta}\widetilde{T}^{(\mathrm{M})}_{\alpha\beta} = \frac{1}{(16\pi\widetilde{G}f)^2} g^{\alpha\beta} T^{(\mathrm{M})}_{\alpha\beta} \tag{4.139}$$

is the trace of the energy-momentum tensor in the conformal frame.

Varying (4.135) with respect to $\tilde{g}_{\mu\nu}$ and ϕ returns equations of motion equivalent to Einstein's equations and an equation for ϕ. The gravitational equation is

$$\widetilde{G}_{\mu\nu} = 8\pi\widetilde{G}\left(\widetilde{T}^{(\mathrm{M})}_{\mu\nu} + \widetilde{T}^{(\phi)}_{\mu\nu}\right), \tag{4.140}$$

where

$$\widetilde{T}^{(\phi)}_{\mu\nu} = \widetilde{\nabla}_\mu\phi\widetilde{\nabla}_\nu\phi - \tilde{g}_{\mu\nu}\left[\tfrac{1}{2}\tilde{g}^{\rho\sigma}\widetilde{\nabla}_\rho\phi\widetilde{\nabla}_\sigma\phi + V(\phi)\right], \tag{4.141}$$

and the scalar field equation is

$$\widetilde{\Box}\phi - \frac{dV}{d\phi} = \frac{1}{2f}\frac{df}{d\phi}\widetilde{T}^{(\mathrm{M})}. \tag{4.142}$$

Given that (4.140) looks just like Einstein's equation with both matter and scalar-field sources, why should we even bother to call this scalar-tensor theory an alternative to GR? Isn't it the same theory, just in different variables? In fact it is not the same, because of the dependence of S_{M} on ϕ in the Einstein frame. In particular, physical test particles will move along geodesics of $g_{\mu\nu}$, which will not generally coincide with those of $\tilde{g}_{\mu\nu}$. The original metric is the one that test particles "see." So either we work in the original variables $(g_{\mu\nu}, \lambda)$, where the gravitational field equation is altered, or we use the new variables $(\tilde{g}_{\mu\nu}, \phi)$, in which the equations of motion for matter are altered; either way, there will be unambiguously measurable departures (in principle) from ordinary GR.

Another way to modify general relativity is to allow for the existence of extra spatial dimensions; in fact the physical consequences of extra dimensions turn out to be closely related to those of scalar-tensor theories. By extra dimensions we don't simply mean considering GR in higher-dimensional spaces, but rather considering models in which the spacetime appears four-dimensional on large scales even though there are really $4 + d$ total dimensions. The simplest way for this to happen is if the extra d dimensions are "compactified" on some manifold; it is this possibility we consider here.[4] Models of this kind are known as Kaluza-Klein theories.

Let G_{ab} be the metric for a $(4 + d)$-dimensional spacetime with coordinates X^a, where indices a, b run from 0 to $d + 3$.

[4]We follow the analysis of S.M. Carroll, J. Geddes, M. Hoffman, and R.M. Wald, *Phys. Rev. D* **66**, 024036 (2002); `http://arxiv.org/hep-th/0110149`. The original papers on extra dimensions are those by Kaluza and Klein: T. Kaluza, *Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.)* **K1**, 966 (1921); O. Klein, *Z. Phys.* **37**, 895 (1926) [*Surveys High Energ. Phys.* **5**, 241 (1926)]; O. Klein, *Nature* **118**, 516 (1926).

$$ds^2 = G_{ab}\mathrm{d}X^a\mathrm{d}X^b = g_{\mu\nu}(x)\mathrm{d}x^\mu\mathrm{d}x^\nu + b^2(x)\gamma_{ij}(y)\mathrm{d}y^i\mathrm{d}y^j, \tag{4.143}$$

where the x^μ are coordinates in the four-dimensional spacetime and the y^i are coordinates on the extra-dimensional manifold, taken to be a maximally symmetric space with metric γ_{ij}. Of course the geometry of the extra dimensions is actually something dynamical that should be determined by solving the full equations of motion, but we are going to take (4.143) as a simplifying ansatz. (In a more complete treatment, we would expand the dynamical modes of the compactified geometry as a Fourier series, and show that the modes we are presently neglecting have larger masses than the overall-size mode we are choosing to examine.) The action is the $(4+d)$-dimensional Hilbert action plus a matter term:

$$S = \int d^{4+d}X\sqrt{-G}\left(\frac{1}{16\pi G_{4+d}}R[G_{ab}] + \widehat{\mathcal{L}}_{\mathrm{M}}\right), \tag{4.144}$$

where $\sqrt{-G}$ is the square root of minus the determinant of G_{ab}, $R[G_{ab}]$ is the Ricci scalar of G_{ab}, and $\widehat{\mathcal{L}}_{\mathrm{M}}$ is the matter Lagrange density with the metric determinant factored out.

The first step is to dimensionally reduce the action (4.144). By this we mean to actually perform the integral over the extra dimensions, which is possible because we have assumed that the extra-dimensional scale factor b is independent of y^i. Therefore we can express everything in terms of $g_{\mu\nu}$, γ_{IJ}, and $b(x)$, integrate over the extra dimensions, and arrive at an effective four-dimensional theory. From the metric (4.143) we have

$$\sqrt{-G} = b^d\sqrt{-g}\sqrt{\gamma}, \tag{4.145}$$

and we can evaluate the curvature scalar for this metric to obtain

$$\begin{aligned} R[G_{ab}] = R[g_{\mu\nu}] + b^{-2}R[\gamma_{ij}] - 2db^{-1}g^{\mu\sigma}\nabla_\mu\nabla_\sigma b \\ - d(d-1)b^{-2}g^{\mu\sigma}(\nabla_\mu b)(\nabla_\sigma b), \end{aligned} \tag{4.146}$$

where ∇_μ is the covariant derivative associated with the four-dimensional metric $g_{\mu\nu}$. We denote by $\mathcal{V}$ the volume of the extra dimensions when $b = 1$; it is given by

$$\mathcal{V} = \int d^d y\sqrt{\gamma}. \tag{4.147}$$

The four-dimensional Newton's constant G_4 is determined by evaluating the coefficient of the curvature scalar in the action; we find that G_4 is related to its higher-dimensional analogue by

$$\frac{1}{16\pi G_4} = \frac{\mathcal{V}}{16\pi G_{4+d}}. \tag{4.148}$$

We are thus left with

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{16\pi G_4}\Big[b^d R[g_{\mu\nu}] + d(d-1)b^{d-2}g^{\mu\nu}(\nabla_\mu b)(\nabla_\nu b) + d(d-1)\kappa b^{d-2}\Big] + \mathcal{V}b^d\widehat{\mathcal{L}}_{\rm M}\right\}, \tag{4.149}$$

where we have integrated by parts for convenience, and introduced the curvature parameter κ of γ_{ij}, given by

$$\kappa = \frac{R[\gamma_{ij}]}{d(d-1)}. \tag{4.150}$$

Comparing to (4.117)–(4.120), we see that the dimensionally-reduced action is precisely that of a scalar-tensor theory; the size of the extra dimensions plays the role of the scalar field. We can therefore make it look more conventional by performing a change of variables and a conformal transformation,

$$\begin{aligned}\beta(x) &= \ln b,\\ \tilde{g}_{\mu\nu} &= e^{d\beta}g_{\mu\nu},\end{aligned} \tag{4.151}$$

which turns the reduced action into that of a scalar field coupled to gravity in the Einstein frame. Following the same procedure as outlined in our discussion of scalar-tensor theories yields

$$S = \int d^4x\sqrt{-\tilde{g}}\left\{\frac{1}{16\pi G_4}\Big[R[\tilde{g}_{\mu\nu}] - \frac{1}{2}d(d+2)\tilde{g}^{\mu\nu}(\widetilde{\nabla}_\mu\beta)(\widetilde{\nabla}_\nu\beta) + d(d-1)\kappa e^{(d+2)\beta}\Big] + \mathcal{V}e^{-d\beta}\widehat{\mathcal{L}}_{\rm M}\right\}, \tag{4.152}$$

where we have dropped terms that are total derivatives.

To turn β into a canonically normalized scalar field, we make one final change of variables, to

$$\phi = \sqrt{\frac{d(d+2)}{2}}\bar{m}_{\rm P}\beta, \tag{4.153}$$

where the reduced Planck mass is $\bar{m}_{\rm P} = (8\pi G_4)^{-1/2}$. We are then left with

$$S = \int d^4x\sqrt{-\tilde{g}}\left\{\frac{1}{16\pi G_4}R[\tilde{g}_{\mu\nu}] - \frac{1}{2}\tilde{g}^{\mu\nu}(\widetilde{\nabla}_\mu\phi)(\widetilde{\nabla}_\nu\phi) + \frac{1}{2}\kappa d(d-1)\bar{m}_{\rm P}^2 e^{-\sqrt{2(d+2)/d}\phi/\bar{m}_{\rm P}} + \mathcal{V}e^{-\sqrt{2d/(d+2)}\phi/\bar{m}_{\rm P}}\widehat{\mathcal{L}}_{\rm M}\right\}. \tag{4.154}$$

The scalar ϕ is known as the **dilaton** or **radion**, and characterizes the size of the extra-dimensional manifold.

The last two terms in (4.154) represent (minus) the potential $V(\phi)$. If we ignore the matter term $\widehat{\mathcal{L}}_{\mathrm{M}}$, the behavior of the dilaton will depend only on the sign of κ. If the extra-dimensional manifold is flat ($\kappa = 0$), the potential vanishes and we simply have a massless scalar field; this possibility runs afoul of the experimental constraints on scalar-tensor theories mentioned above. If there is curvature ($\kappa \neq 0$), the potential has no minimum; for $\kappa > 0$ the field will roll to $-\infty$, while for $\kappa < 0$ the field will roll to $+\infty$. But $\phi \propto \ln b$, so this means the scale factor $b(x)$ of the extra dimensions either shrinks to zero or becomes arbitrarily large, in either case ruining the hope for stable extra dimensions. Stability can be achieved, however, by choosing an appropriate matter Lagrangian, and an appropriate field configuration in the extra dimensions.

Let us now move on to a different kind of alternative theory, those that feature Lagrangians of more than second order in derivatives of the metric. We could imagine an action of the form

$$S = \int d^n x \sqrt{-g}(R + \alpha_1 R^2 + \alpha_2 R_{\mu\nu} R^{\mu\nu} + \alpha_3 g^{\mu\nu} \nabla_\mu R \nabla_\nu R + \cdots), \quad (4.155)$$

where the α's are coupling constants and the dots represent every other scalar we can make from the curvature tensor, its contractions, and its derivatives. Traditionally, such terms have been neglected on the reasonable grounds that they merely complicate a theory that is already both aesthetically pleasing and empirically successful. There is also, classically speaking, a more substantive objection. In conventional form, Einstein's equation leads to a well-posed initial value problem for the metric, in which coordinates and momenta specified at an initial time can be used to predict future evolution. With higher-derivative terms, we would require not only those data, but also some number of derivatives of the momenta; the character of the theory is dramatically altered.

However, there are also good reasons to consider such additional terms. As mentioned in our brief discussion of quantum gravity, one of the technical obstacles to consistent quantization of general relativity is that the theory is nonrenormalizable: Inclusion of higher-order quantum effects leads to infinite answers. With the appropriate combination of higher-order Lagrangian terms, it turns out that you can actually render the theory renormalizable, which gives some hope of constructing a consistent quantum theory.[5] Unfortunately, it turns out that renormalizability comes at too high a price; these models generally feature negative-energy field excitations (ghosts). Consequently, the purported vacuum state (empty space) would be unstable to decay into positive- and negative-energy modes, which is inconsistent with both empirical experience and theoretical prejudice.

Nevertheless, the prevailing current view is that GR is an effective theory valid at energies below the Planck scale, and we should actually include all of the pos-

[5] See, for example, K.S. Stelle, *Phys. Rev.* **D16**, 953 (1977).

sible higher-order terms; but they will be suppressed by appropriate powers of the Planck scale, just as we argued in our discussion of the Equivalence Principle in Section 4.7. They will therefore only become important when the length scale characteristic of the curvature approaches the Planck scale, which is far from any plausible experiment. Higher-order terms are therefore interesting in principle, but not in practice. On the other hand, similar reasoning would lead us to expect a huge vacuum energy term, since it is lower-order than the Hilbert action, which we know not to be true; so we should keep an open mind.

As a final alternative to general relativity, we should mention the possibility that the connection really is not derived from the metric, but in fact has an independent existence as a fundamental field. As one of the exercises you are asked to show that it is possible to consider the conventional action for general relativity but treat it as a function of both the metric $g_{\mu\nu}$ and a torsion-free connection $\Gamma^{\lambda}_{\rho\sigma}$, and the equations of motion derived from varying such an action with respect to the connection imply that $\Gamma^{\lambda}_{\rho\sigma}$ is actually the Christoffel connection associated with $g_{\mu\nu}$. We could drop the demand that the connection be torsion-free, in which case the torsion tensor could lead to additional propagating degrees of freedom. The basic reason why such theories do not receive much attention is simply because the torsion is itself a tensor; there is nothing to distinguish it from other, nongravitational tensor fields. Thus, we do not really lose any generality by considering theories of torsion-free connections (which lead to GR) plus any number of tensor fields, which we can name what we like. Similar considerations apply when we consider dropping the requirement of metric compatibility—any connection can be written as a metric-compatible connection plus a tensorial correction, so any such theory is equivalent to GR plus extra tensor fields, which wouldn't really deserve to be called an "alternative to general relativity".

4.9 ■ EXERCISES

1. The Lagrange density for electromagnetism in curved space is

$$\mathcal{L} = \sqrt{-g}\left(-\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + A_\mu J^\mu\right). \tag{4.156}$$

where J^μ is the conserved current.

(a) Derive the energy-momentum tensor by functional differentiation with respect to the metric. You can assume that the $A_\mu J^\mu$ term does not contribute to the energy-momentum tensor.

(b) Consider adding a new term to the Lagrangian,

$$\mathcal{L}' = \beta R^{\mu\nu} g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma}.$$

How are Maxwell's equations altered in the presence of this term? Einstein's equation? Is the current still conserved?

2. We showed how to derive Einstein's equation by varying the Hilbert action with respect to the metric. They can also be derived by treating the metric and connection as independent degrees of freedom and varying separately with respect to them; this is known

as the **Palatini formalism**. That is, we consider the action

$$S = \int d^4x \sqrt{-g}\, g^{\mu\nu} R_{\mu\nu}(\Gamma),$$

where the Ricci tensor is thought of as constructed purely from the connection, not using the metric. Variation with respect to the metric gives the usual Einstein's equations, but for a Ricci tensor constructed from a connection that has no a priori relationship to the metric. Imagining from the start that the connection is symmetric (torsion free), show that variation of this action with respect to the connection coefficients leads to the requirement that the connection be metric compatible, that is, the Christoffel connection. Remember that Stokes's theorem, relating the integral of the covariant divergence of a vector to an integral of the vector over the boundary, does not work for a general covariant derivative. The best strategy is to write the connection coefficients as a sum of the Christoffel symbols $\widetilde{\Gamma}^{\lambda}_{\mu\nu}$ and a tensor $C^{\lambda}{}_{\mu\nu}$,

$$\Gamma^{\lambda}_{\mu\nu} = \widetilde{\Gamma}^{\lambda}_{\mu\nu} + C^{\lambda}{}_{\mu\nu},$$

and then show that $C^{\lambda}{}_{\mu\nu}$ must vanish.

3. The four-dimensional δ-function on a manifold M is defined by

$$\int_M F(x^\mu) \left[\frac{\delta^{(4)}(x^\sigma - y^\sigma)}{\sqrt{-g}} \right] \sqrt{-g}\, d^4x = F(y^\sigma), \tag{4.157}$$

for an arbitrary function $F(x^\mu)$. Meanwhile, the energy-momentum tensor for a pressureless perfect fluid (dust) is

$$T^{\mu\nu} = \rho U^\mu U^\nu, \tag{4.158}$$

where ρ is the energy density and U^μ is the four-velocity. Consider such a fluid that consists of a single particle traveling on a world line $x^\mu(\tau)$, with τ the proper time. The energy-momentum tensor for this fluid is then given by

$$T^{\mu\nu}(y^\sigma) = m \int_M \left[\frac{\delta^{(4)}(y^\sigma - x^\sigma(\tau))}{\sqrt{-g}} \right] \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} d\tau, \tag{4.159}$$

where m is the rest mass of the particle. Show that covariant conservation of the energy-momentum tensor, $\nabla_\mu T^{\mu\nu} = 0$, implies that $x^\mu(\tau)$ satisfies the geodesic equation.

4. Show that the energy-momentum tensors for electromagnetism and for scalar field theory satisfy the dominant energy condition, and thus also the weak, null, and null dominant conditions. Show that they also satisfy $w \geq -1$.

5. A spacetime is static if there is a timelike Killing vector that is orthogonal to spacelike hypersurfaces. (See Appendices D and F for more discussion, including a definition of Raychaudhuri's equation.)

   (a) Generally speaking, if a vector field v^μ is orthogonal to a set of hypersurfaces defined by $f =$ constant, then we can write the vector as $v_\mu = h\nabla_\mu f$ (here both f and h are functions). Show that this implies

$$v_{[\sigma} \nabla_\mu v_{\nu]} = 0.$$

(b) Imagine we have a perfect fluid with zero pressure (dust), which generates a solution to Einstein's equations. Show that the metric can be static only if the fluid four-velocity is parallel to the timelike (and hypersurface-orthogonal) Killing vector.

(c) Use Raychaudhuri's equation to prove that there is no static solution to Einstein's equations if the pressure is zero and the energy density is greater than zero.

6. Let K be a Killing vector field. Show that an electromagnetic field with potential $A_\mu = K_\mu$ solves Maxwell's equations if the metric is a vacuum solution to Einstein's equations. This is a slight cheat, since you won't be in vacuum if there is a nonzero electromagnetic field strength, but we assume the field strength is small enough not to dramatically affect the geometry.

CHAPTER

5 The Schwarzschild Solution

5.1 ■ THE SCHWARZSCHILD METRIC

The most obvious application of a theory of gravity is to a spherically symmetric gravitational field. This would be the relevant situation to describe, for example, the field created by the Earth or the Sun (to a good approximation), in which apples fall or planets move. Furthermore, our first concern is with exterior solutions (empty space surrounding a gravitating body), since understanding the motion of test particles outside an object is both easier and more immediately useful than considering the relatively inaccessible interior. In addition to its practical usefulness, the answer to this problem in general relativity will lead us to remarkable solutions describing new phenomena of great interest to physicists and astronomers: black holes. In this chapter we examine the simple case of vacuum solutions with perfect spherical symmetry; in the next chapter we consider features of black holes in more general contexts.

In GR, the unique spherically symmetric vacuum solution is the **Schwarzschild metric**; it is second only to Minkowski space in the list of important spacetimes. In spherical coordinates $\{t, r, \theta, \phi\}$, the metric is given by

$$\boxed{ds^2 = -\left(1 - \frac{2GM}{r}\right) \mathrm{d}t^2 + \left(1 - \frac{2GM}{r}\right)^{-1} \mathrm{d}r^2 + r^2\, d\Omega^2,} \tag{5.1}$$

where $d\Omega^2$ is the metric on a unit two-sphere,

$$d\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2. \tag{5.2}$$

The constant M is interpreted as the mass of the gravitating object (although some care is required in making this identification). In this section we will derive the Schwarzschild metric by trial and error; in the next section we will be more systematic in both the derivation of the solution and its consequences.

Since we are interested in the solution *outside* a spherical body, we care about Einstein's equation in vacuum,

$$R_{\mu\nu} = 0. \tag{5.3}$$

Our hypothesized source is static (unevolving) and spherically symmetric, so we will look for solutions that also have these properties. Rigorous definitions of both "static" and "spherically symmetric" require some care, due to subtleties of coordinate independence. For now we will interpret static to imply two conditions: that all metric components are independent of the time coordinate, and that there are no time-space cross terms ($dt\,dx^i + dx^i\,dt$) in the metric. The latter condition makes sense if we imagine performing a time inversion $t \to -t$; the dt^2 term remains invariant, as do any $dx^i dx^j$ terms, while cross terms would not. Since we hope to find a solution that is independent of time, it should be invariant under time reversal, and we therefore leave cross terms out. To impose spherical symmetry, we begin by writing the metric of Minkowski space (a spherically symmetric spacetime we know something about) in polar coordinates $x^\mu = (t, r, \theta, \phi)$:

$$ds^2_{\text{Minkowski}} = -dt^2 + dr^2 + r^2\,d\Omega^2. \tag{5.4}$$

One requirement to preserve spherical symmetry is that we maintain the form of $d\Omega^2$; that is, if we want our spheres to be perfectly round, the coefficient of the $d\phi^2$ term should be $\sin^2\theta$ times that of the $d\theta^2$ term. But we are otherwise free to multiply all of the terms by separate coefficients, so long as they are only functions of the radial coordinate r:

$$ds^2 = -e^{2\alpha(r)}\,dt^2 + e^{2\beta(r)}\,dr^2 + e^{2\gamma(r)}r^2\,d\Omega^2. \tag{5.5}$$

We've expressed our functions as exponentials so that the signature of the metric doesn't change. In a full treatment, we would allow for complete freedom and see what happens.

We can use our ability to change coordinates to make a slight simplification to the static, spherically-symmetric metric (5.5), even before imposing Einstein's equation. Unlike other theories of physics, in general relativity we simultaneously define coordinates and the metric as a function of those coordinates. In other words, we don't know ahead of time what, for example, the radial coordinate r really is; we can only interpret it once the solution is in our hands. Let us therefore imagine defining a new coordinate $\bar{r}$ via

$$\bar{r} = e^{\gamma(r)}r, \tag{5.6}$$

with an associated basis one-form

$$d\bar{r} = e^{\gamma}\,dr + e^{\gamma}r\,d\gamma = \left(1 + r\frac{d\gamma}{dr}\right)e^{\gamma}\,dr. \tag{5.7}$$

In terms of this new variable, the metric (5.5) becomes

$$ds^2 = -e^{2\alpha(r)}\,dt^2 + \left(1 + r\frac{d\gamma}{dr}\right)^{-2} e^{2\beta(r)-2\gamma(r)}d\bar{r}^2 + \bar{r}^2\,d\Omega^2, \tag{5.8}$$

where each function of r is a function of $\bar{r}$ in the obvious way. But now let us make the following relabelings:

$$\bar{r} \to r \tag{5.9}$$

$$\left(1 + r\frac{d\gamma}{dr}\right)^{-2} e^{2\beta(r)-2\gamma(r)} \to e^{2\bar{\beta}}. \tag{5.10}$$

There is nothing to stop us from doing this, as they are simply labels, with no independent external definition. If you wish you can continue to use $\bar{r}$, and set (5.10) equal to $e^{2\bar{\beta}}$, but we won't bother. Our metric (5.8) becomes

$$ds^2 = -e^{2\alpha(r)}\,\mathrm{d}t^2 + e^{2\beta(r)}\,\mathrm{d}r^2 + r^2\,d\Omega^2. \tag{5.11}$$

This looks exactly like (5.5), except that the $e^{2\gamma}$ factor has disappeared. We have not set $e^{2\gamma}$ equal to one, which would be a statement about the geometry; we have simply chosen our radial coordinate such that this factor doesn't exist. Thus, (5.11) is precisely as general as (5.5).

Let's now take this metric and use Einstein's equation to solve for the functions $\alpha(r)$ and $\beta(r)$. We begin by evaluating the Christoffel symbols. If we use labels (t, r, θ, ϕ) for $(0, 1, 2, 3)$ in the usual way, the Christoffel symbols are given by

$$\begin{aligned}
&\Gamma^t_{tr} = \partial_r\alpha && \Gamma^r_{tt} = e^{2(\alpha-\beta)}\partial_r\alpha && \Gamma^r_{rr} = \partial_r\beta \\
&\Gamma^\theta_{r\theta} = \frac{1}{r} && \Gamma^r_{\theta\theta} = -re^{-2\beta} && \Gamma^\phi_{r\phi} = \frac{1}{r} \\
&\Gamma^r_{\phi\phi} = -re^{-2\beta}\sin^2\theta && \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta && \Gamma^\phi_{\theta\phi} = \frac{\cos\theta}{\sin\theta}.
\end{aligned} \tag{5.12}$$

Anything not written down explicitly is meant to be zero, or related to what is written by symmetries. From these we get the following nonvanishing components of the Riemann tensor:

$$\begin{aligned}
R^t{}_{rtr} &= \partial_r\alpha\partial_r\beta - \partial_r^2\alpha - (\partial_r\alpha)^2 \\
R^t{}_{\theta t\theta} &= -re^{-2\beta}\partial_r\alpha \\
R^t{}_{\phi t\phi} &= -re^{-2\beta}\sin^2\theta\,\partial_r\alpha \\
R^r{}_{\theta r\theta} &= re^{-2\beta}\partial_r\beta \\
R^r{}_{\phi r\phi} &= re^{-2\beta}\sin^2\theta\,\partial_r\beta \\
R^\theta{}_{\phi\theta\phi} &= (1 - e^{-2\beta})\sin^2\theta.
\end{aligned} \tag{5.13}$$

Taking the contraction as usual yields the Ricci tensor:

$$\begin{aligned}
R_{tt} &= e^{2(\alpha-\beta)}\left[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \tfrac{2}{r}\partial_r\alpha\right] \\
R_{rr} &= -\partial_r^2\alpha - (\partial_r\alpha)^2 + \partial_r\alpha\partial_r\beta + \tfrac{2}{r}\partial_r\beta \\
R_{\theta\theta} &= e^{-2\beta}[r(\partial_r\beta - \partial_r\alpha) - 1] + 1 \\
R_{\phi\phi} &= \sin^2\theta\,R_{\theta\theta},
\end{aligned} \tag{5.14}$$

and for future reference we calculate the curvature scalar,

$$R = -2e^{-2\beta}\left[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \frac{2}{r}(\partial_r\alpha - \partial_r\beta) + \frac{1}{r^2}(1 - e^{2\beta})\right]. \tag{5.15}$$

With the Ricci tensor calculated, we would like to set it equal to zero. Since R_{tt} and R_{rr} vanish independently, we can write

$$0 = e^{2(\beta-\alpha)}R_{tt} + R_{rr} = \frac{2}{r}(\partial_r\alpha + \partial_r\beta), \tag{5.16}$$

which implies $\alpha = -\beta + c$, where c is some constant. We can set this constant equal to zero by rescaling our time coordinate by $t \to e^{-c}t$, after which we have

$$\alpha = -\beta. \tag{5.17}$$

Next let us turn to $R_{\theta\theta} = 0$, which now reads

$$e^{2\alpha}(2r\partial_r\alpha + 1) = 1. \tag{5.18}$$

This is equivalent to

$$\partial_r(re^{2\alpha}) = 1. \tag{5.19}$$

We can solve this to obtain

$$e^{2\alpha} = 1 - \frac{R_S}{r}, \tag{5.20}$$

where R_S is some undetermined constant. With (5.17) and (5.20), our metric becomes

$$ds^2 = -\left(1 - \frac{R_S}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{R_S}{r}\right)^{-1}\mathrm{d}r^2 + r^2\,d\Omega^2. \tag{5.21}$$

We now have no freedom left except for the single constant R_S, so this form had better solve the remaining equations $R_{tt} = 0$ and $R_{rr} = 0$; it is straightforward to check that it does, for any value of R_S.

The only thing left to do is to interpret the constant R_S, called the **Schwarzschild radius**, in terms of some physical parameter. Nothing could be simpler. In Chapter 4 we found that, in the weak-field limit, the tt component of the metric around a point mass satisfies

$$g_{tt} = -\left(1 - \frac{2GM}{r}\right). \tag{5.22}$$

The Schwarzschild metric should reduce to the weak-field case when $r \gg 2GM$, but for the tt component the forms are already exactly the same; we need only identify

$$R_S = 2GM. \tag{5.23}$$

This can be thought of as the definition of the parameter M.

Our final result is the Schwarzschild metric, (5.1). We have shown that it is a static, spherically symmetric vacuum solution to Einstein's equation; M functions as a parameter, which we happen to know can be interpreted as the conventional Newtonian mass that we would measure by studying orbits at large distances from the gravitating source. It won't simply be the sum of the masses of the constituents of whatever body is curving spacetime, since there will be a contribution from what we might think of as the gravitational binding energy; however, in the weak field limit, the quantities will agree. Note that as $M \to 0$ we recover Minkowski space, which is to be expected. Note also that the metric becomes progressively Minkowskian as $r \to \infty$; this property is known as **asymptotic flatness**. A more technical definition involves matching regions at infinity in a conformal diagram, as discussed in the next chapter.

5.2 ■ BIRKHOFF'S THEOREM

Birkhoff's theorem is the statement that the Schwarzschild metric is the *unique* vacuum solution with spherical symmetry (and in particular, that there are no time-dependent solutions of this form); proving it is an instructive exercise, which consists of three major steps. First, we argue that a spherically symmetric spacetime can be foliated by two-spheres—in other words, that (almost) every point lies on a unique sphere that is left invariant by the generators of spherical symmetry. Second, we show on purely geometric grounds that the metric on such a space can always (at least in a local region) be put in the form

$$ds^2 = d\tau^2(a, b) + r^2(a, b)\, d\Omega^2(\theta, \phi), \tag{5.24}$$

where (a, b) are coordinates transverse to the spheres, and r is a function of these coordinates. Third, we plug this metric into Einstein's equation in vacuum to show that Schwarzschild is the unique solution. We will argue in favor of the first two points at a level of rigor that is likely to be convincing to most physicists, although mathematicians will be uneasy; the third point is straightforward calculation. For a more careful treatment see Hawking and Ellis (1973). We will use a few concepts from Appendix C, which may be useful to read at this point. Of course, if you are more interested in exploring properties of the Schwarzschild solution than in proving its uniqueness, you are welcome to skip right to the next section.

We begin with the concept of a four-dimensional spherically symmetric spacetime M. Spherically symmetric means having the same symmetries as a sphere. (In this chapter the word sphere refers specifically to S^2, not spheres of other dimension.) The symmetries of a sphere are precisely those of ordinary rotations in three-dimensional Euclidean space; in the language of group theory, they comprise the special orthogonal group SO(3). (Recall the discussion of the Lorentz and rotation groups in Chapter 1.) In the case of a metric on a manifold, symmetries are characterized by the existence of Killing vectors. In Section 3.8 we found the three Killing vectors of S^2, labeled (R, S, T); in (θ, ϕ) coordinates they take

the form

$$R = \partial_\phi$$
$$S = \cos\phi\,\partial_\theta - \cot\theta\sin\phi\,\partial_\phi$$
$$T = -\sin\phi\,\partial_\theta - \cot\theta\cos\phi\,\partial_\phi. \tag{5.25}$$

A spherically symmetric manifold is one that has three Killing vector fields that are the same as those on S^2. But how do we know, in a coordinate-independent way, that a set of Killing vectors on one manifold is the same as that on some other manifold? The structure of a set of symmetry transformations is given by the commutation relations of the transformations, which express the difference between performing two infinitesimal transformations in one order versus the reversed order. In group theory these are expressed by the Lie algebra of the symmetry generators, while in differential geometry they are expressed by the commutators of the Killing vector fields. There is a deep connection here, which we don't have time to pursue; see Schutz (1980). In the Exercises for Chapter 3 you verified that the commutators of the rotational Killing vectors (R, S, T) satisfied

$$[R, S] = T$$
$$[S, T] = R$$
$$[T, R] = S. \tag{5.26}$$

This algebra of Killing vectors fully characterizes the kind of symmetry we have. A manifold will be said to possess **spherical symmetry** if and only if there are three Killing fields satisfying (5.26).

In Appendix C we discuss Frobenius's theorem, which states that if you have a set of vector fields whose commutator closes—the commutator of any two fields in the set is a linear combination of other fields in the set—then the integral curves of these vector fields fit together to describe submanifolds of the manifold on which they are all defined. The dimensionality of the submanifold may be smaller than the number of vectors, or it could be equal, but obviously not larger. Vector fields that obey (5.26) will of course form 2-spheres. Since the vector fields stretch throughout the space, every point will be on exactly one of these spheres. (Actually, it's almost every point—we will show below how it can fail to be absolutely every point.) Thus, we say that a spherically symmetric manifold can be foliated into spheres.

Let's consider some examples to bring this down to earth. The simplest example is flat three-dimensional Euclidean space. If we pick an origin, then $\mathbf{R}^3$ is clearly spherically symmetric with respect to rotations around this origin. Under such rotations (that is, under the flow of the Killing vector fields), points move into each other, but each point stays on an S^2 at a fixed distance from the origin.

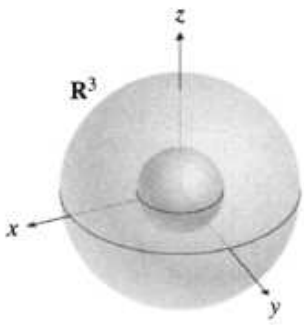


FIGURE 5.1 Foliating $\mathbf{R}^3$ (minus the origin) by two-spheres.

These spheres foliate $\mathbf{R}^3$, as depicted in Figure 5.1. Of course, they don't really foliate all of the space, since the origin itself just stays put under rotations—it

FIGURE 5.2 Foliation of a wormhole by two-spheres.

doesn't move around on some two-sphere. But it should be clear that almost all of the space is properly foliated, and this will turn out to be enough for us.

We can also have spherical symmetry without an origin to rotate things around. An example is provided by a wormhole, with topology $\mathbf{R} \times S^2$. If we suppress a dimension and draw our two-spheres as circles, such a space might look like Figure 5.2. In this case the entire manifold can be foliated by two-spheres.

Given that manifolds with SO(3) symmetry may be foliated by spheres, our second step is to show that the metric on M can be put into the form (5.24). The set of all the spheres forms a two-dimensional space (since a four-dimensional spacetime is being foliated with two-dimensional spheres). You might hope we could simply put coordinates (θ, ϕ) on each sphere, and coordinates (a, b) on the set of all spheres, for a complete set of coordinates (a, b, θ, ϕ) on M. Then each sphere is specified by a = constant, b = constant. We know that the metric on a round sphere is $d\Omega^2$, so this strategy would be sufficient to guarantee that the metric restricted to any fixed values $a = a_0$ and $b = b_0$ (so that $\mathrm{d}a = \mathrm{d}b = 0$) takes the form

$$ds^2(a_0, b_0, \theta, \phi) = f(a_0, b_0)\, d\Omega^2. \tag{5.27}$$

In particular, the function f must be independent of θ and ϕ, or the sphere would be lumpy rather than round. Furthermore, it's equally clear that the metric restricted to any fixed values $\theta = \theta_0$ and $\phi = \phi_0$ (so that $\mathrm{d}\theta = \mathrm{d}\phi = 0$) takes the form

$$ds^2(a, b, \theta_0, \phi_0) = d\tau^2(a, b). \tag{5.28}$$

Again, any dependence on θ or ϕ would destroy the symmetry; it would mean that the geometry transverse to the spheres depended on where you were on the sphere.

However, we have been too reckless by slapping down these coordinates, since we cannot rule out cross terms of the form $\mathrm{d}a\mathrm{d}\theta + \mathrm{d}\theta\mathrm{d}a$ and so on. In other words, we must be careful to line up our spheres appropriately, so that travel along a curve that is perpendicular to one of the spheres keeps us at constant θ and ϕ. To guarantee this we need to be more careful in setting up our coordinates. Begin by considering a single point q lying on a sphere S_q (note that q must not be a degenerate point at which all of the Killing vectors vanish). Put coordinates (θ, ϕ) on this particular sphere only, not yet through the manifold. At each point p on S_q, there will be a two-dimensional orthogonal subspace O_p, consisting of points along geodesics emanating from p whose tangent vectors at p are orthogonal to S_q. Note that there will be a one-dimensional subgroup R_p of rotations that leave p fixed; indeed, these rotations keep fixed any direction perpendicular to S_q at p, and hence the entire two-surface O_p is left invariant by R_p.

Consider a point r that is not on S_q, but on some other sphere S_r in the foliation, and that lies in the two-surface O_p orthogonal to S_q at p. Since p is arbitrary, this includes any possible point r in a neighborhood of S_q. Note that O_p will be orthogonal to S_r as well as to S_q. To see this, consider the two-dimensional plane

V_r of vectors in the tangent space $T_r M$ that are orthogonal to the two-surface O_p. Since O_p is left invariant by the rotations R_p, these rotations must take V_r into itself, because they are an isometry, and hence preserve orthogonality. But R_p also takes the set of vectors tangent to S_q into itself, since these rotations leave the spheres invariant. In four dimensions, two planes that are both orthogonal to a given plane at the same point must be the same plane; hence, the vectors tangent to S_r must be orthogonal to O_p.

There will be a unique geodesic that is orthogonal to S_q and connects p to r. Traveling down such geodesics provides a map $f : S_q \to S_r$, which is both one-to-one and onto (at least in a neighborhood of the original sphere). We use this map to define coordinates on S_r (and, similarly, on any other sphere) by assigning the same values of (θ, ϕ) to $r \in S_r$ that were the coordinates at $p \in S_q$. We have therefore defined (θ, ϕ) throughout the manifold. Now to define coordinates (a, b), choose two basis vectors S, T for the subspace of $T_q M$ that generates the orthogonal space O_q. Any other sphere will be connected to q by a unique orthogonal geodesic, with tangent vector $aS + bT \in T_q M$. Assign those components (a, b) as coordinates everywhere on that sphere. This defines the full set of coordinates (a, b, θ, ϕ) throughout the manifold.

The metric in these coordinates satisfies (5.27) and (5.28); it remains to be shown that there are no cross terms between directions along the spheres and those transverse. This means, for example, that the vector field ∂_a should be orthogonal to ∂_θ, and so on; it is straightforward to verify that this is so. First, consider ∂_θ at some point $r \in S_r$; this vector is the directional derivative along a curve of the form $x^\mu(\theta) = (a_r, b_r, \theta, \phi_r)$. Since a and b are constant along the curve, the entire curve remains in the sphere S_r, so that ∂_θ is tangent to the sphere. Meanwhile, ∂_a is a derivative along $x^\mu(a) = (a, b_r, \theta_r, \phi_r)$. Since this curve remains in the orthogonal subspace O_r, ∂_a will be orthogonal to S_r, and hence to ∂_θ. Similar arguments guarantee that there will be no cross terms between (a, b) and (θ, ϕ).

We have thus succeeded in putting the metric on a spherically symmetric spacetime in the form

$$ds^2 = g_{aa}(a, b)\, \mathrm{d}a^2 + g_{ab}(a, b)(\mathrm{d}a\mathrm{d}b + \mathrm{d}b\mathrm{d}a) + g_{bb}(a, b)\, \mathrm{d}b^2 + r^2(a, b)\, d\Omega^2. \tag{5.29}$$

Here $r(a, b)$ is some as-yet-undetermined function, to which we have merely given a suggestive label. There is nothing to stop us, however, from changing coordinates from (a, b) to (a, r) by inverting $r(a, b)$, unless r were a function of a alone; in this case we could just as easily switch to (b, r), so we will not consider this situation separately. The metric is then

$$ds^2 = g_{aa}(a, r)\, \mathrm{d}a^2 + g_{ar}(a, r)(\mathrm{d}a\, \mathrm{d}r + \mathrm{d}r\, \mathrm{d}a) + g_{rr}(a, r)\, \mathrm{d}r^2 + r^2\, d\Omega^2. \tag{5.30}$$

Our next step is to find a function $t(a, r)$ such that, in the (t, r) coordinate system, there are no cross terms $\mathrm{d}t\mathrm{d}r + \mathrm{d}r\mathrm{d}t$ in the metric. Notice that

$$\mathrm{d}t = \frac{\partial t}{\partial a}\, \mathrm{d}a + \frac{\partial t}{\partial r}\, \mathrm{d}r, \tag{5.31}$$

so

$$\mathrm{d}t^2 = \left(\frac{\partial t}{\partial a}\right)^2 \mathrm{d}a^2 + \left(\frac{\partial t}{\partial a}\right)\left(\frac{\partial t}{\partial r}\right)(\mathrm{d}a\,\mathrm{d}r + \mathrm{d}r\,\mathrm{d}a) + \left(\frac{\partial t}{\partial r}\right)^2 \mathrm{d}r^2. \tag{5.32}$$

We would like to replace the first three terms in the metric (5.30) by

$$m\mathrm{d}t^2 + n\mathrm{d}r^2, \tag{5.33}$$

for some functions m and n. This is equivalent to the requirements

$$m\left(\frac{\partial t}{\partial a}\right)^2 = g_{aa}, \tag{5.34}$$

$$n + m\left(\frac{\partial t}{\partial r}\right)^2 = g_{rr}, \tag{5.35}$$

and

$$m\left(\frac{\partial t}{\partial a}\right)\left(\frac{\partial t}{\partial r}\right) = g_{ar}. \tag{5.36}$$

We therefore have three equations for the three unknowns $t(a, r)$, $m(a, r)$, and $n(a, r)$, just enough to determine them precisely, up to initial conditions for t. (Of course, they are "determined" in terms of the unknown functions g_{aa}, g_{ar}, and g_{rr}, so in this sense they are still undetermined.) We can therefore put our metric in the form

$$ds^2 = m(t, r)\,\mathrm{d}t^2 + n(t, r)\,\mathrm{d}r^2 + r^2\,d\Omega^2. \tag{5.37}$$

To this point the only difference between the two coordinates t and r is that we have chosen r to be the one that multiplies the metric for the two-sphere. This choice was motivated by what we know about the metric for flat Minkowski space, which can be written $ds^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\,d\Omega^2$. We know that the spacetime under consideration is Lorentzian, so either m or n will have to be negative. Let us choose m, the coefficient of $\mathrm{d}t^2$, to be negative. This is not a choice we are simply allowed to make, and in fact we will see later that it can go wrong; but we will assume it for now. The assumption is not completely unreasonable, since we know that Minkowski space is itself spherically symmetric, and will therefore be described by (5.37). With this choice we can trade in the functions m and n for new functions α and β, such that

$$ds^2 = -e^{2\alpha(t,r)}\,\mathrm{d}t^2 + e^{2\beta(t,r)}\,\mathrm{d}r^2 + r^2\,d\Omega^2. \tag{5.38}$$

This is the best we can do using only geometry; spherical symmetry is certainly not enough to say anything substantive about the functions $\alpha(t, r)$ and $\beta(t, r)$. Our next step is therefore to actually solve Einstein's equation; the steps follow closely

along those of Section 5.1, in which we considered a metric similar to (5.38) but with the additional assumption of time-independence. Here we will see that this assumption was unnecessary, as the solution will necessarily be static.

The nonvanishing Christoffel symbols for (5.38) are

$$\begin{aligned}
&\Gamma^t_{tt} = \partial_t\alpha && \Gamma^t_{tr} = \partial_r\alpha && \Gamma^t_{rr} = e^{2(\beta-\alpha)}\partial_t\beta \\
&\Gamma^r_{tt} = e^{2(\alpha-\beta)}\partial_r\alpha && \Gamma^r_{tr} = \partial_t\beta && \Gamma^r_{rr} = \partial_r\beta \\
&\Gamma^\theta_{r\theta} = \frac{1}{r} && \Gamma^r_{\theta\theta} = -re^{-2\beta} && \Gamma^\phi_{r\phi} = \frac{1}{r} \\
&\Gamma^r_{\phi\phi} = -re^{-2\beta}\sin^2\theta && \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta && \Gamma^\phi_{\theta\phi} = \frac{\cos\theta}{\sin\theta},
\end{aligned} \tag{5.39}$$

the nonvanishing components of the Riemann tensor are

$$\begin{aligned}
R^t{}_{rtr} &= e^{2(\beta-\alpha)}[\partial_t^2\beta + (\partial_t\beta)^2 - \partial_t\alpha\partial_t\beta] + [\partial_r\alpha\partial_r\beta - \partial_r^2\alpha - (\partial_r\alpha)^2] \\
R^t{}_{\theta t\theta} &= -re^{-2\beta}\partial_r\alpha \\
R^t{}_{\phi t\phi} &= -re^{-2\beta}\sin^2\theta\,\partial_r\alpha \\
R^t{}_{\theta r\theta} &= -re^{-2\alpha}\partial_t\beta \\
R^t{}_{\phi r\phi} &= -re^{-2\alpha}\sin^2\theta\,\partial_t\beta \\
R^r{}_{\theta r\theta} &= re^{-2\beta}\partial_r\beta \\
R^r{}_{\phi r\phi} &= re^{-2\beta}\sin^2\theta\,\partial_r\beta \\
R^\theta{}_{\phi\theta\phi} &= (1-e^{-2\beta})\sin^2\theta,
\end{aligned} \tag{5.40}$$

and the Ricci tensor is

$$\begin{aligned}
R_{tt} &= \left[\partial_t^2\beta + (\partial_t\beta)^2 - \partial_t\alpha\partial_t\beta\right] + e^{2(\alpha-\beta)}\left[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \frac{2}{r}\partial_r\alpha\right] \\
R_{rr} &= -\left[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta - \frac{2}{r}\partial_r\beta\right] \\
&\quad + e^{2(\beta-\alpha)}\left[\partial_t^2\beta + (\partial_t\beta)^2 - \partial_t\alpha\partial_t\beta\right] \\
R_{tr} &= \frac{2}{r}\partial_t\beta \\
R_{\theta\theta} &= e^{-2\beta}\left[r(\partial_r\beta - \partial_r\alpha) - 1\right] + 1 \\
R_{\phi\phi} &= R_{\theta\theta}\sin^2\theta.
\end{aligned} \tag{5.41}$$

Our job is to solve Einstein's equation in vacuum, $R_{\mu\nu} = 0$. From $R_{tr} = 0$ we get

$$\partial_t\beta = 0. \tag{5.42}$$

If we consider taking the time derivative of $R_{\theta\theta} = 0$ and using $\partial_t \beta = 0$, we get

$$\partial_t \partial_r \alpha = 0. \tag{5.43}$$

We can therefore write

$$\begin{aligned} \beta &= \beta(r) \\ \alpha &= f(r) + g(t). \end{aligned} \tag{5.44}$$

The first term in the metric (5.38) is thus $-e^{2f(r)}e^{2g(t)}\,dt^2$. But we can always simply redefine our time coordinate by replacing $dt \to e^{-g(t)}\,dt$; in other words, we are free to choose t such that $g(t) = 0$, whence $\alpha(t, r) = f(r)$. We therefore have

$$ds^2 = -e^{2\alpha(r)}\,dt^2 + e^{2\beta(r)}\,dr^2 + r^2\,d\Omega^2. \tag{5.45}$$

All of the metric components are independent of the coordinate t. We have therefore proven a crucial result: *any spherically symmetric vacuum metric possesses a timelike Killing vector.*

This property is so interesting that it gets its own name: a metric that possesses a Killing vector that is timelike near infinity is called **stationary**. (Often, including in Schwarzschild, the Killing vector that is timelike at infinity will become spacelike somewhere in the interior.) In a stationary metric we can choose coordinates (t, x^1, x^2, x^3) in which the Killing vector is ∂_t and the metric components are independent of t; the general form of a stationary metric in these coordinates is thus

$$ds^2 = g_{00}(\vec{x})\,dt^2 + g_{0i}(\vec{x})(dt\,dx^i + dx^i\,dt) + g_{ij}(\vec{x})\,dx^i dx^j. \tag{5.46}$$

There is also a more restrictive property: a metric is called **static** if it possesses a timelike Killing vector that is orthogonal to a family of hypersurfaces. (For more details on hypersurfaces, see Appendix D.) In the Exercises for Chapter 4 you showed that a hypersurface-orthogonal vector field v^μ obeys

$$v_{[\mu}\nabla_\nu v_{\sigma]} = 0. \tag{5.47}$$

But there is a simpler diagnostic; if we have adapted coordinates so that the components $g_{\mu\nu}$ are all independent of t, the surfaces to which the Killing vector will be orthogonal are defined by the condition $t =$ constant. Operationally, this means that the time-space cross terms in (5.46) will be absent; the general static metric can be written

$$ds^2 = g_{00}(\vec{x})\,dt^2 + g_{ij}(\vec{x})\,dx^i dx^j. \tag{5.48}$$

We notice that only even powers of the time coordinate t appear in this form; thus, an alternative definition of "static" is "stationary, and invariant under time reversal ($t \to -t$)." The metric (5.45) is clearly static. You should think of stationary as meaning "doing exactly the same thing at every time," while static means "not

doing anything at all." For example, the static spherically symmetric metric (5.45) will describe nonrotating stars or black holes, while rotating systems that keep rotating in the same way at all times will be described by metrics that are stationary but not static.

Notice that (5.45) is precisely the same as (5.11), the metric we originally used to derive the Schwarzschild solution in Section 5.1. We have therefore proven Birkhoff's theorem, that the unique spherically symmetric vacuum solution is the Schwarzschild metric,

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) \mathrm{d}t^2 + \left(1 - \frac{2GM}{r}\right)^{-1} \mathrm{d}r^2 + r^2 \, d\Omega^2, \qquad (5.49)$$

as promised.

We did not say anything about the source of the Schwarzschild metric, except that it be spherically symmetric. Specifically, we did not demand that the source itself be static; it could be a collapsing star, as long as the collapse is symmetric. Therefore a process such as a supernova explosion would generate very little gravitational radiation (in comparison to the amount of energy released through other channels) if it were close to spherically symmetric, which a realistic supernova may or may not be depending on its origin. This is the same result we would have obtained in electromagnetism, where the electromagnetic fields around a spherical charge distribution do not depend on the radial distribution of the charges.

5.3 ■ SINGULARITIES

Before exploring the behavior of test particles in the Schwarzschild geometry, we should say something about singularities. From the form of (5.1), the metric coefficients become infinite at $r = 0$ and $r = 2GM$—an apparent sign that something is going wrong. The metric coefficients, of course, are coordinate-dependent quantities, and as such we should not make too much of their values; it is certainly possible to have a coordinate singularity that results from a breakdown of a specific coordinate system rather than the underlying manifold. An example occurs at the origin of polar coordinates in the plane, where the metric $ds^2 = \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2$ becomes degenerate and the component $g^{\theta\theta} = r^{-2}$ of the inverse metric blows up, even though that point of the manifold is no different from any other.

What kind of coordinate-independent signal should we look for as a warning that something about the geometry is out of control? This turns out to be a difficult question to answer, and entire books have been written about the nature of singularities in general relativity. We won't go into this issue in detail, but rather turn to one simple criterion for when something has gone wrong—when the curvature becomes infinite. The curvature is measured by the Riemann tensor, and it is hard to say when a tensor becomes infinite, since its components are coordinate-dependent. But from the curvature we can construct various scalar quantities, and since scalars are coordinate-independent it is meaningful to say that they become infinite. The simplest such scalar is the Ricci scalar

$R = g^{\mu\nu}R_{\mu\nu}$, but we can also construct higher-order scalars such as $R^{\mu\nu}R_{\mu\nu}$, $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$, $R_{\mu\nu\rho\sigma}R^{\rho\sigma\lambda\tau}R_{\lambda\tau}{}^{\mu\nu}$, and so on. If any of these scalars (but not necessarily all of them) goes to infinity as we approach some point, we regard that point as a singularity of the curvature. We should also check that the point is not infinitely far away; that is, that it can be reached by traveling a finite distance along a curve.

We therefore have a sufficient condition for a point to be considered a singularity. It is not a necessary condition, however, and it is generally harder to show that a given point is nonsingular; for our purposes we will simply test to see if geodesics are well-behaved at the point in question, and if so then we will consider the point nonsingular. In the case of the Schwarzschild metric (5.1), direct calculation reveals that

$$R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = \frac{48G^2M^2}{r^6}. \tag{5.50}$$

This is enough to convince us that $r = 0$ represents an honest singularity.

The other trouble spot is $r = 2GM$, the Schwarzschild radius. You could check that none of the curvature invariants blows up there. We therefore begin to think that it is actually not singular, and we have simply chosen a bad coordinate system. The best thing to do is to transform to more appropriate coordinates if possible. We will soon see that in this case it is in fact possible, and the surface $r = 2GM$ is very well-behaved (although interesting) in the Schwarzschild metric—it demarcates the event horizon of a black hole.

Having worried a little about singularities, we should point out that the behavior of the Schwarzschild metric inside the Schwarzschild radius is of little day-to-day consequence. The solution we derived is valid only in vacuum, and we expect it to hold outside a spherical body such as a star. However, in the case of the Sun we are dealing with a body that extends to a radius of

$$R_\odot = 10^6 GM_\odot. \tag{5.51}$$

Thus, $r = 2GM_\odot$ is far inside the solar interior, where we do not expect the Schwarzschild metric to apply. In fact, realistic stellar interior solutions consist of matching the exterior Schwarzschild metric to an interior metric that is perfectly smooth at the origin. Nevertheless, there are objects for which the full Schwarzschild metric is required—black holes—and therefore we will let our imaginations roam far outside the solar system in this chapter.

5.4 ■ GEODESICS OF SCHWARZSCHILD

The first step we will take to understand the Schwarzschild metric more fully is to consider the behavior of geodesics. We need the nonzero Christoffel symbols for Schwarzschild:

$$\Gamma^r_{tt} = \frac{GM}{r^3}(r-2GM) \qquad \Gamma^r_{rr} = \frac{-GM}{r(r-2GM)} \qquad \Gamma^t_{tr} = \frac{GM}{r(r-2GM)}$$
$$\Gamma^\theta_{r\theta} = \frac{1}{r} \qquad \Gamma^r_{\theta\theta} = -(r-2GM) \qquad \Gamma^\phi_{r\phi} = \frac{1}{r}$$
$$\Gamma^r_{\phi\phi} = -(r-2GM)\sin^2\theta \qquad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta \qquad \Gamma^\phi_{\theta\phi} = \frac{\cos\theta}{\sin\theta}. \tag{5.52}$$

The geodesic equation therefore turns into the following four equations, where λ is an affine parameter:

$$\frac{d^2t}{d\lambda^2} + \frac{2GM}{r(r-2GM)}\frac{dr}{d\lambda}\frac{dt}{d\lambda} = 0,$$
$$\frac{d^2r}{d\lambda^2} + \frac{GM}{r^3}(r-2GM)\left(\frac{dt}{d\lambda}\right)^2 - \frac{GM}{r(r-2GM)}\left(\frac{dr}{d\lambda}\right)^2 - (r-2GM)\left[\left(\frac{d\theta}{d\lambda}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\lambda}\right)^2\right] = 0,$$
$$\frac{d^2\theta}{d\lambda^2} + \frac{2}{r}\frac{d\theta}{d\lambda}\frac{dr}{d\lambda} - \sin\theta\cos\theta\left(\frac{d\phi}{d\lambda}\right)^2 = 0,$$
$$\frac{d^2\phi}{d\lambda^2} + \frac{2}{r}\frac{d\phi}{d\lambda}\frac{dr}{d\lambda} + 2\frac{\cos\theta}{\sin\theta}\frac{d\theta}{d\lambda}\frac{d\phi}{d\lambda} = 0. \tag{5.53}$$

There does not seem to be much hope for simply solving this set of coupled equations by inspection. Fortunately our task is greatly simplified by the high degree of symmetry of the Schwarzschild metric. We know that there are four Killing vectors: three for the spherical symmetry, and one for time translations. Each of these will lead to a constant of the motion for a free particle. If K^μ is a Killing vector, we know that

$$K_\mu \frac{dx^\mu}{d\lambda} = \text{constant}. \tag{5.54}$$

In addition, we always have another constant of the motion for geodesics: the geodesic equation (together with metric compatibility) implies that the quantity

$$\epsilon = -g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} \tag{5.55}$$

is constant along the path. (For any trajectory we can choose the parameter λ such that ϵ is a constant; we are simply noting that this is compatible with affine parameterization along a geodesic.) Of course, for a massive particle we typically choose $\lambda = \tau$, and this relation simply becomes $\epsilon = -g_{\mu\nu}U^\mu U^\nu = +1$. For massless particles, which move along null trajectories, we always have $\epsilon = 0$,

and this equation does not fix the parameter λ. As discussed in Section 3.4, it is convenient to normalize λ along null geodesics such that the four-momentum and four-velocity are equal, $p^\mu = dx^\mu/d\lambda$. We might also be concerned with spacelike geodesics (even though they do not correspond to paths of particles), for which we will choose $\epsilon = -1$.

Rather than immediately writing out explicit expressions for the four conserved quantities associated with Killing vectors, let's think about what they are telling us. Notice that the symmetries they represent are also present in flat spacetime, where the conserved quantities they lead to are very familiar. Invariance under time translations leads to conservation of energy, while invariance under spatial rotations leads to conservation of the three components of angular momentum. Essentially the same applies to the Schwarzschild metric. We can think of the angular momentum as a three-vector with a magnitude (one component) and direction (two components). Conservation of the direction of angular momentum means that the particle will move in a plane. We can choose this to be the equatorial plane of our coordinate system; if the particle is not in this plane, we can rotate coordinates until it is. Thus, the two Killing vectors that lead to conservation of the direction of angular momentum imply that, for a single particle, we can choose

$$\theta = \frac{\pi}{2}. \tag{5.56}$$

The two remaining Killing vectors correspond to energy and the magnitude of angular momentum. The energy arises from the timelike Killing vector

$$K^\mu = (\partial_t)^\mu = (1, 0, 0, 0). \tag{5.57}$$

The Killing vector whose conserved quantity is the magnitude of the angular momentum is

$$R^\mu = (\partial_\phi)^\mu = (0, 0, 0, 1). \tag{5.58}$$

In both cases it is convenient to lower the index to obtain

$$K_\mu = \left(-\left(1 - \frac{2GM}{r}\right), 0, 0, 0\right) \tag{5.59}$$

and

$$R_\mu = \left(0, 0, 0, r^2 \sin^2\theta\right). \tag{5.60}$$

Since (5.56) implies that $\sin\theta = 1$ along the geodesics of interest to us, the two conserved quantities are

$$E = -K_\mu \frac{dx^\mu}{d\lambda} = \left(1 - \frac{2GM}{r}\right)\frac{dt}{d\lambda} \tag{5.61}$$

and

$$L = R_\mu \frac{dx^\mu}{d\lambda} = r^2 \frac{d\phi}{d\lambda}. \tag{5.62}$$

For massless particles, these can be thought of as the conserved energy and angular momentum, while for massive particles they are the conserved energy and angular momentum per unit mass of the particle. In the discussion of rotating black holes in the next chapter, we will use E and L to refer to the actual energy and angular momentum, not "per unit mass"; the meaning should be clear from context. Note that the constancy of (5.62) is the GR equivalent of Kepler's second law—equal areas are swept out in equal times.

Recall that in Section 3.4 we claimed that the energy of a particle with four-momentum p^μ, as measured by an observer with four-velocity U^μ, would be $-p_\mu U^\mu$. This is *not* equal, or even proportional, to (5.61), even if the observer is taken to be stationary ($U^i = 0$). Mathematically, this is because the four-velocity is normalized to $U_\mu U^\mu = -1$, which the Killing vector K^μ is not: If we tried to normalize it in that way, it would no longer solve Killing's equation. At a slightly deeper level, $-p_\mu U^\mu$ may be thought of as the inertial/kinetic energy of the particle, while $-p_\mu K^\mu$ is the total conserved energy, including the potential energy due to the gravitational field. The notion of gravitational potential energy is not always well-defined, but the total energy is well-defined in the presence of a timelike Killing vector. We will presently use E to help characterize geodesics of Schwarzschild; later we will also use $-p_\mu U^\mu$ for massless particles, where it can be thought of as the observed frequency of a photon, to describe gravitational redshift.

Together the conserved quantities E and L provide a convenient way to understand the orbits of particles in the Schwarzschild geometry. Let us expand the expression (5.55) for ϵ to obtain

$$-\left(1 - \frac{2GM}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\phi}{d\lambda}\right)^2 = -\epsilon. \tag{5.63}$$

If we multiply this by $(1 - 2GM/r)$ and use our expressions for E and L, we obtain

$$-E^2 + \left(\frac{dr}{d\lambda}\right)^2 + \left(1 - \frac{2GM}{r}\right)\left(\frac{L^2}{r^2} + \epsilon\right) = 0. \tag{5.64}$$

This is certainly progress, since we have taken a messy system of coupled equations and obtained a single equation for $r(\lambda)$. It looks even nicer if we rewrite it as

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + V(r) = \mathcal{E}, \tag{5.65}$$

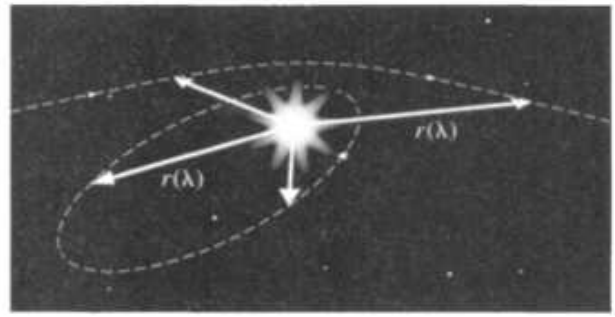


FIGURE 5.3 Orbits around a star are characterized by giving the radius r as a function of a parameter λ.

where

$$V(r) = \frac{1}{2}\epsilon - \epsilon\frac{GM}{r} + \frac{L^2}{2r^2} - \frac{GML^2}{r^3} \tag{5.66}$$

and

$$\mathcal{E} = \frac{1}{2}E^2. \tag{5.67}$$

In (5.65) we have precisely the equation for a classical particle of unit mass and "energy" $\mathcal{E}$ moving in a one-dimensional potential given by $V(r)$. It's a little confusing, but not too bad: the conserved energy per unit mass is E, but the effective potential for the coordinate r responds to $\mathcal{E} = E^2/2$.

Of course, our physical situation is quite different from a classical particle moving in one dimension; the trajectories under consideration are orbits around a star or other object, as shown in Figure 5.3. The quantities of interest to us are not only $r(\lambda)$, but also $t(\lambda)$ and $\phi(\lambda)$. Nevertheless, we can go a long way toward understanding all of the orbits by understanding their radial behavior, and it is a great help to reduce this behavior to a problem we know how to solve.

A similar analysis of orbits in Newtonian gravity would have produced a similar result; the general equation (5.65) would have been the same, but the effective potential (5.66) would not have had the last term. (Note that this equation is not a power series in $1/r$, it is exact.) In the potential (5.66) the first term is just a constant, the second term corresponds exactly to the Newtonian gravitational potential, and the third term is a contribution from angular momentum that takes the same form in Newtonian gravity and general relativity. The last term, the GR contribution, will turn out to make a great deal of difference, especially at small r.

Let us examine the effective potentials for different kinds of possible orbits, as illustrated in Figures 5.4 and 5.5. There are different curves $V(r)$ for different values of L; for any one of these curves, the behavior of the orbit can be judged by comparing $\mathcal{E}$ to $V(r)$. The general behavior of the particle will be to move in the potential until it reaches a "turning point" where $V(r) = \mathcal{E}$, when it will begin

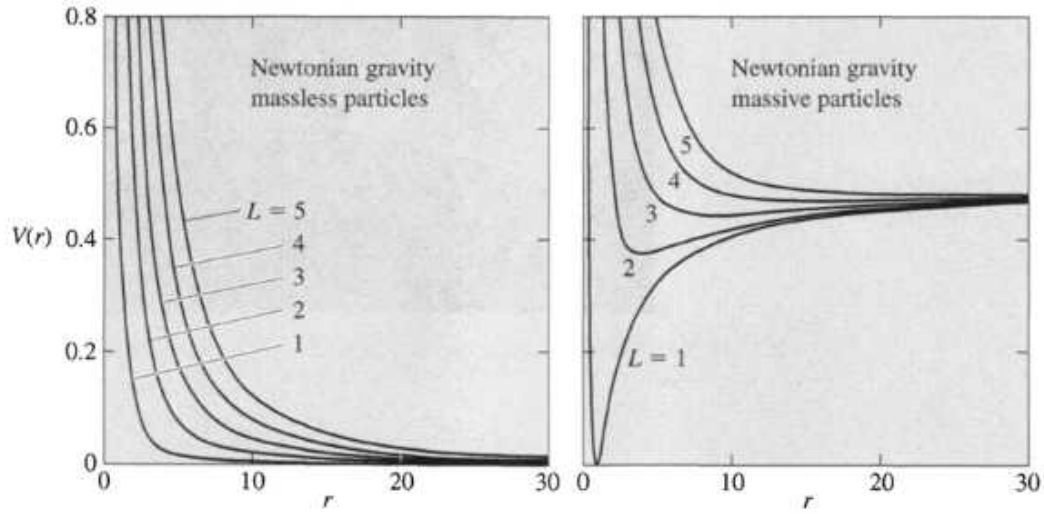


FIGURE 5.4 Effective potentials in Newtonian gravity. Five curves are shown, corresponding to the listed values of the angular momentum (per unit mass) L, and we have chosen $GM = 1$. Note that, for large enough energy, every orbit reaches a turning point and returns to infinity.

moving in the other direction. Sometimes there may be no turning point to hit, in which case the particle just keeps going. In other cases the particle may simply move in a circular orbit at radius r_c = constant; this can happen at points where the potential is flat, $dV/dr = 0$. Differentiating (5.66), we find that the circular orbits occur when

$$\epsilon GMr_c^2 - L^2 r_c + 3GML^2\gamma = 0, \tag{5.68}$$

where $\gamma = 0$ in Newtonian gravity and $\gamma = 1$ in general relativity. Circular orbits will be stable if they correspond to a minimum of the potential, and unstable if they correspond to a maximum. Bound orbits that are not circular will oscillate around the radius of the stable circular orbit.

Turning to Newtonian gravity, we find that circular orbits appear at

$$r_c = \frac{L^2}{\epsilon GM}. \tag{5.69}$$

For massless particles, $\epsilon = 0$, and there are no circular orbits; this is consistent with the first plot in Figure 5.4, which illustrates that there are no bound orbits of any sort. Although it is somewhat obscured in polar coordinates, massless particles actually move in a straight line, since the Newtonian gravitational force on a massless particle is zero. Of course the standing of massless particles in Newtonian theory is somewhat problematic, so you can get different answers depending on what assumptions you make. In terms of the effective potential, a photon with a given energy E will come in from $r = \infty$ and gradually slow down (actually $dr/d\lambda$ will decrease, but the speed of light isn't changing) until it reaches the

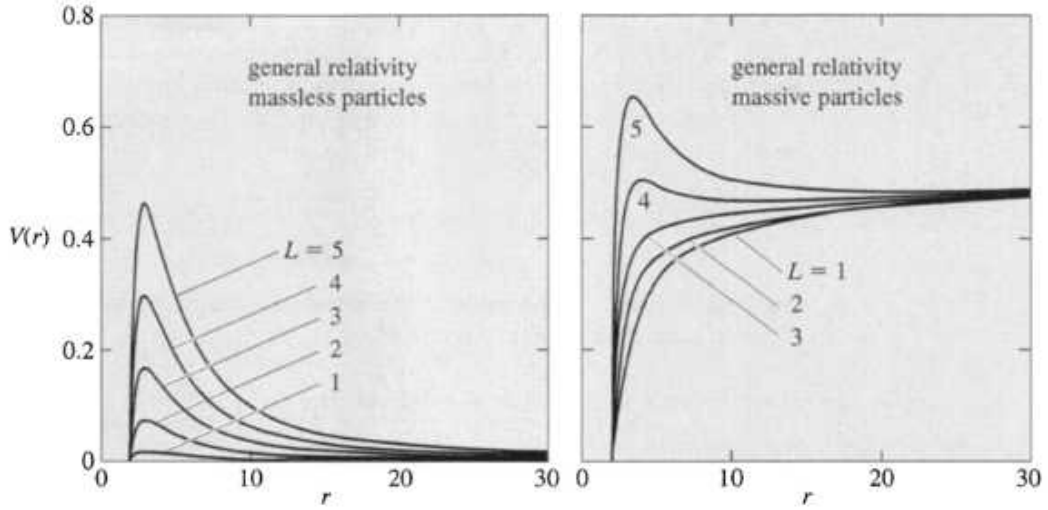


FIGURE 5.5 Effective potentials in general relativity. Again, five curves are shown, corresponding to the listed values of the angular momentum (per unit mass) L, and we have chosen $GM = 1$. In GR there is an innermost circular orbit greater than or equal to $3GM$, and any orbit that falls inside this radius continues to $r = 0$ (for particles on geodesics).

turning point, when it will start moving away back to $r = \infty$. The lower values of L, for which the photon will come closer before it starts moving away, are simply those trajectories that are initially aimed closer to the gravitating body. For massive particles there will be stable circular orbits at the radius (5.69), as well as bound orbits that oscillate around this radius. If the energy is greater than the asymptotic value $E = 1$, the orbits will be unbound, describing a particle that approaches the star and then recedes. We know that the orbits in Newton's theory are conic sections—bound orbits are either circles or ellipses, while unbound ones are either parabolas or hyperbolas—although we won't show that here.

In general relativity the situation is different, but only for r sufficiently small. Since the difference resides in the term $-GML^2/r^3$, as $r \to \infty$ the behaviors are identical in the two theories. But as $r \to 0$ the potential goes to $-\infty$ rather than $+\infty$ as in the Newtonian case. At $r = 2GM$ the potential is always zero; inside this radius is the black hole, which we will discuss more thoroughly later. For massless particles there is always a barrier (except for $L = 0$, for which the potential vanishes identically), but a sufficiently energetic photon will nevertheless go over the barrier and be dragged inexorably down to the center. Note that "sufficiently energetic" means "in comparison to its angular momentum"—in fact the frequency of the photon is immaterial, only the direction in which it is pointing. At the top of the barrier are unstable circular orbits. For $\epsilon = 0$, $\gamma = 1$, we can easily solve (5.68) to obtain

$$r_c = 3GM. \tag{5.70}$$

This is borne out by the first part of Figure 5.5, which shows a maximum of $V(r)$ at $r = 3GM$ for every L. This means that a photon can orbit forever in a circle at this radius, but any perturbation will cause it to fly away either to $r = 0$ or $r = \infty$.

For massive particles there are once again different regimes depending on the angular momentum. The circular orbits are at

$$r_c = \frac{L^2 \pm \sqrt{L^4 - 12G^2M^2L^2}}{2GM}. \tag{5.71}$$

For large L there will be two circular orbits, one stable and one unstable. In the $L \to \infty$ limit their radii are given by

$$r_c = \frac{L^2 \pm L^2(1 - 6G^2M^2/L^2)}{2GM} = \left(\frac{L^2}{GM},\ 3GM\right). \tag{5.72}$$

In this limit the stable circular orbit becomes farther away, while the unstable one approaches $3GM$, behavior that parallels the massless case. As we decrease L, the two circular orbits come closer together; they coincide when the discriminant in (5.71) vanishes, which is at

$$L = \sqrt{12}GM, \tag{5.73}$$

for which

$$r_c = 6GM, \tag{5.74}$$

and they disappear entirely for smaller L. Thus $6GM$ is the smallest possible radius of a stable circular orbit in the Schwarzschild metric. There are also unbound orbits, which come in from infinity and turn around, and bound but noncircular orbits, which oscillate around the stable circular radius. Note that such orbits, which would describe exact conic sections in Newtonian gravity, will not do so in GR, although we would have to solve the equation for $d\phi/d\lambda$ to demonstrate it. Finally, there are orbits that come in from infinity and continue all the way in to $r = 0$; this can happen either if the energy is higher than the barrier, or for $L < \sqrt{12}GM$, when the barrier goes away entirely.

We have therefore found that the Schwarzschild solution possesses stable circular orbits for $r > 6GM$ and unstable circular orbits for $3GM < r < 6GM$. It's important to remember that these are only the geodesics; there is nothing to stop an accelerating particle from dipping below $r = 3GM$ and emerging, as long as it stays beyond $r = 2GM$.

5.5 ■ EXPERIMENTAL TESTS

Most experimental tests of general relativity involve the motion of test particles in the solar system, and hence geodesics of the Schwarzschild metric. Einstein

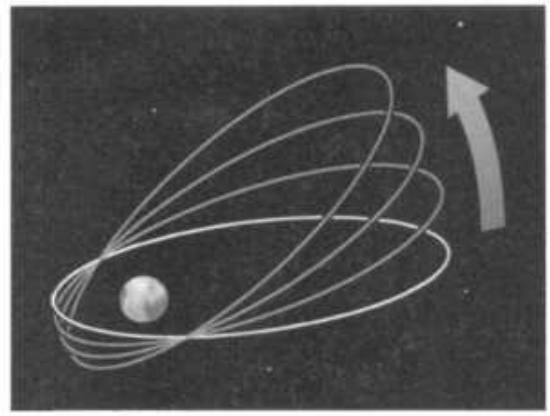


FIGURE 5.6 Orbits in general relativity describe precessing ellipses.

suggested three tests: the deflection of light, the precession of perihelia, and gravitational redshift. The deflection of light is observable in the weak-field limit, and is therefore discussed in Chapter 7. In this section we will discuss the precession of perihelia and the gravitational redshift. (The perihelion of an elliptical orbit is its point of closest approach to the Sun; orbits around the Earth or a star would have perigee or periastron, respectively.)

The precession of perihelia reflects the fact that noncircular orbits in GR are not perfect closed ellipses; to a good approximation they are ellipses that precess, describing a flower pattern as shown in Figure 5.6. Despite its conceptual simplicity, the rate of perihelion precession is somewhat cumbersome to calculate; here we follow d'Inverno (1992). The strategy is to describe the evolution of the radial coordinate r as a function of the angular coordinate ϕ; for a perfect ellipse, $r(\phi)$ would be periodic with period 2π, reflecting the fact that perihelion occurred at the same angular position each orbit. Using perturbation theory we can show how GR introduces a slight alteration of the period, giving rise to precession.

We start with our radial equation of motion of a massive particle in a Schwarzschild metric (5.65). To get an equation for $dr/d\phi$ we multiply by

$$\left(\frac{d\phi}{d\lambda}\right)^{-2} = \frac{r^4}{L^2}, \tag{5.75}$$

which yields

$$\left(\frac{dr}{d\phi}\right)^2 + \frac{1}{L^2}r^4 - \frac{2GM}{L^2}r^3 + r^2 - 2GMr = \frac{2\mathcal{E}}{L^2}r^4. \tag{5.76}$$

Two tricks are useful in solving this equation. The first trick is to define a new variable

$$x = \frac{L^2}{GMr}. \tag{5.77}$$

From (5.69) we see that $x = 1$ at a Newtonian circular orbit. Our equation of motion (5.76) becomes

$$\left(\frac{dx}{d\phi}\right)^2 + \frac{L^2}{G^2M^2} - 2x + x^2 - \frac{2G^2M^2}{L^2}x^3 = \frac{2\mathcal{E}L^2}{G^2M^2}. \tag{5.78}$$

The second trick is to differentiate this with respect to ϕ, obtaining a second-order equation for $x(\phi)$:

$$\frac{d^2x}{d\phi^2} - 1 + x = \frac{3G^2M^2}{L^2}x^2. \tag{5.79}$$

In a Newtonian calculation, the last term would be absent, and we could solve for x exactly; here, we can treat it as a perturbation.

We expand x into a Newtonian solution plus a small deviation,

$$x = x_0 + x_1. \tag{5.80}$$

The zeroth-order part of (5.79) is then

$$\frac{d^2x_0}{d\phi^2} - 1 + x_0 = 0 \tag{5.81}$$

and the first-order part is

$$\frac{d^2x_1}{d\phi^2} + x_1 = \frac{3G^2M^2}{L^2}x_0^2. \tag{5.82}$$

The solution for the zeroth-order equation can be written

$$x_0 = 1 + e\cos\phi. \tag{5.83}$$

This is the standard result of Newton or Kepler; it describes a perfect ellipse, with e the eccentricity. An ellipse is specified by the semi-major axis a, the distance from the center to the farthest point on the ellipse, and the semi-minor axis b, the distance from the center to the closest point. The eccentricity satisfies $e^2 = 1 - b^2/a^2$.

Plugging the Newtonian solution into the first-order equation (5.82), we obtain

$$\begin{aligned}\frac{d^2x_1}{d\phi^2} + x_1 &= \frac{3G^2M^2}{L^2}(1 + e\cos\phi)^2 \\ &= \frac{3G^2M^2}{L^2}\left[\left(1 + \frac{1}{2}e^2\right) + 2e\cos\phi + \frac{1}{2}e^2\cos 2\phi\right].\end{aligned} \tag{5.84}$$

To solve this equation, notice that

$$\frac{d^2}{d\phi^2}(\phi \sin\phi) + \phi \sin\phi = 2\cos\phi \tag{5.85}$$

and

$$\frac{d^2}{d\phi^2}(\cos 2\phi) + \cos 2\phi = -3\cos 2\phi. \tag{5.86}$$

Comparing these to (5.84), we see that a solution is provided by

$$x_1 = \frac{3G^2M^2}{L^2}\left[\left(1 + \frac{1}{2}e^2\right) + e\phi\sin\phi - \frac{1}{6}e^2\cos 2\phi\right], \tag{5.87}$$

as you are welcome to check. The three terms here have different characters. The first is simply a constant displacement, while the third oscillates around zero. The important effect is thus contained in the second term, which accumulates over successive orbits. We therefore combine this term with the zeroth-order solution to write

$$x = 1 + e\cos\phi + \frac{3G^2M^2e}{L^2}\phi\sin\phi. \tag{5.88}$$

This is not a full solution, even to the perturbed equation, but it encapsulates the part that we care about. In particular, this expression for x can be conveniently rewritten as the equation for an ellipse with an angular period that is not quite 2π:

$$x = 1 + e\cos[(1-\alpha)\phi], \tag{5.89}$$

where we have introduced

$$\alpha = \frac{3G^2M^2}{L^2}. \tag{5.90}$$

The equivalence of (5.88) and (5.89) can be seen by expanding $\cos[(1-\alpha)\phi]$ as a power series in the small parameter α:

$$\begin{aligned}\cos[(1-\alpha)\phi] &= \cos\phi + \alpha\frac{d}{d\alpha}\cos[(1-\alpha)\phi]_{\alpha=0} \\ &= \cos\phi + \alpha\phi\sin\phi.\end{aligned} \tag{5.91}$$

We have therefore found that, during each orbit of the planet, perihelion advances by an angle

$$\Delta\phi = 2\pi\alpha = \frac{6\pi G^2M^2}{L^2}. \tag{5.92}$$

To convert from the angular momentum L to more conventional quantities, we may use expressions valid for Newtonian orbits, since the quantity we're looking

at is already a small perturbation. An ordinary ellipse satisfies

$$r = \frac{(1 - e^2)a}{1 + e\cos\phi}, \tag{5.93}$$

where a is the semi-major axis. Comparing to our zeroth-order solution (5.83) and the definition (5.77) of x, we see that

$$L^2 \approx GM(1 - e^2)a. \tag{5.94}$$

This is an approximation, valid if the orbit were a perfect closed ellipse. Plugging this into (5.92) and restoring explicit factors of the speed of light, we obtain

$$\Delta\phi = \frac{6\pi GM}{c^2(1 - e^2)a}. \tag{5.95}$$

Historically, the precession of Mercury was the first test of GR. In fact it was known before Einstein invented GR that there was an apparent discrepancy in Mercury's orbit, and a number of solutions had been proposed (including "dark matter" in the inner Solar System). Einstein knew of the discrepancy, and one of his first tasks after formulating GR was to show that it correctly accounted for Mercury's perihelion precession. For the motion of Mercury around the Sun, the relevant orbital parameters are

$$\begin{aligned} \frac{GM_\odot}{c^2} &= 1.48 \times 10^5 \text{ cm}, \\ a &= 5.79 \times 10^{12} \text{ cm} \\ e &= 0.2056, \end{aligned} \tag{5.96}$$

and of course $c = 3.00 \times 10^{10}$ cm/sec. This gives

$$\Delta\phi_{\text{Mercury}} = 5.01 \times 10^{-7} \text{ radians/orbit} = 0.103''/\text{orbit}, \tag{5.97}$$

where $''$ stands for arcseconds. It is more conventional to express this in terms of precession per century; Mercury orbits once every 88 days, yielding

$$\Delta\phi_{\text{Mercury}} = 43.0''/\text{century}. \tag{5.98}$$

So the major axis of Mercury's orbit precesses at a rate of 43.0 arcsecs every 100 years. The observed value is 5601 arcsecs/100 years. However, much of that is due to the precession of equinoxes in our geocentric coordinate system; 5025 arcsecs/100 years, to be precise. The gravitational perturbations of the other planets contribute an additional 532 arcsecs/100 years, leaving 43 arcsecs/100 years to be explained by GR, which it does quite well. You can imagine that Einstein must have been very pleased when he first figured this out.

In Chapter 2 we discussed the gravitational redshift of photons as a consequence of the Principle of Equivalence. The Schwarzschild metric is an exact

solution of GR, and should therefore predict a redshift that reduces to the EP prediction in small regions of spacetime. Let's see how that works.

Consider an observer with four-velocity U^μ, who is stationary in the Schwarzschild coordinates ($U^i = 0$). We could allow the observer to be moving, but that would merely superimpose a conventional Doppler shift over the gravitational effect. The four-velocity satisfies $U_\mu U^\mu = -1$, which for a stationary observer in Schwarzschild implies

$$U^0 = \left(1 - \frac{2GM}{r}\right)^{-1/2}. \tag{5.99}$$

Any such observer measures the frequency of a photon following along a null geodesic $x^\mu(\lambda)$ to be

$$\omega = -g_{\mu\nu} U^\mu \frac{dx^\nu}{d\lambda}. \tag{5.100}$$

Indeed, this relation defines the normalization of λ. We therefore have

$$\omega = \left(1 - \frac{2GM}{r}\right)^{1/2} \frac{dt}{d\lambda} \tag{5.101}$$

$$= \left(1 - \frac{2GM}{r}\right)^{-1/2} E, \tag{5.102}$$

where E is defined by (5.61), applied to the photon trajectory. E is conserved, so ω will clearly take on different values when measured at different radial distances. For a photon emitted at r_1 and observed at r_2, the observed frequencies will be related by

$$\frac{\omega_2}{\omega_1} = \left(\frac{1 - 2GM/r_1}{1 - 2GM/r_2}\right)^{1/2}. \tag{5.103}$$

This is an exact result for the frequency shift; in the limit $r \gg 2GM$ we have

$$\frac{\omega_2}{\omega_1} = 1 - \frac{GM}{r_1} + \frac{GM}{r_2}$$

$$= 1 + \Phi_1 - \Phi_2, \tag{5.104}$$

where $\Phi = -GM/r$ is the Newtonian potential. This tells us that the frequency goes down as Φ increases, which happens as we climb out of a gravitational field: thus, a redshift. (Photons that fall toward a gravitating body are blueshifted.) We see that the $r \gg 2GM$ result agrees with the calculation based on the Equivalence Principle.

The gravitational redshift was first detected in 1960 by Pound and Rebka, using gamma rays traveling upward a distance of only 72 feet (the height of the physics building at Harvard). Subsequent tests have become increasingly precise, often

making use of artificial spacecraft or atomic clocks carried aboard airplanes. The agreement with Einstein's predictions has been excellent in all cases.

Since Einstein's proposal of the three classic tests, further tests of GR have been proposed. The most famous is of course the binary pulsar, to be discussed in Chapter 7. Another is the gravitational time delay, discovered and observed by Shapiro, also discussed in Chapter 7. In a very different context, Big-Bang nucleosynthesis provides a cosmological test of GR at an epoch when the universe was only seconds old, as discussed in Chapter 8. Modern advances have also introduced a host of new tests; for a comprehensive introduction see Will (1981).

5.6 ■ SCHWARZSCHILD BLACK HOLES

We now know something about the behavior of geodesics outside the troublesome radius $r = 2GM$, which is the regime of interest for the solar system and most other astrophysical situations. We next turn to the study of objects that are described by the Schwarzschild solution even at radii smaller than $2GM$—black holes. (We'll use the term "black hole" for the moment, even though we haven't introduced a precise meaning for such an object.)

One way to understand the geometry of a spacetime is to explore its causal structure, as defined by the light cones. We therefore consider radial null curves, those for which θ and ϕ are constant and $ds^2 = 0$:

$$ds^2 = 0 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2, \tag{5.105}$$

from which we see that

$$\frac{dt}{dr} = \pm\left(1 - \frac{2GM}{r}\right)^{-1}. \tag{5.106}$$

This of course measures the slope of the light cones on a spacetime diagram of the t-r plane. For large r the slope is ± 1, as it would be in flat space, while as we approach $r = 2GM$ we get $dt/dr \to \pm\infty$, and the light cones "close up," as shown in Figure 5.7. Thus a light ray that approaches $r = 2GM$ never seems to get there, at least in this coordinate system; instead it seems to asymptote to this radius.

As we will see, the apparent inability to get to $r = 2GM$ is an illusion, and the light ray (or a massive particle) actually has no trouble reaching this radius. But an observer far away would never be able to tell. If we stayed outside while an intrepid observational general relativist dove into the black hole, sending back signals all the time, we would simply see the signals reach us more and more slowly, as portrayed in Figure 5.8. In the Exercises you are asked to look at this phenomenon more carefully. As an infalling observer approaches $r = 2GM$, any fixed interval $\Delta\tau_1$ of their proper time corresponds to a longer and longer interval $\Delta\tau_2$ from our point of view. This continues forever; we would never see

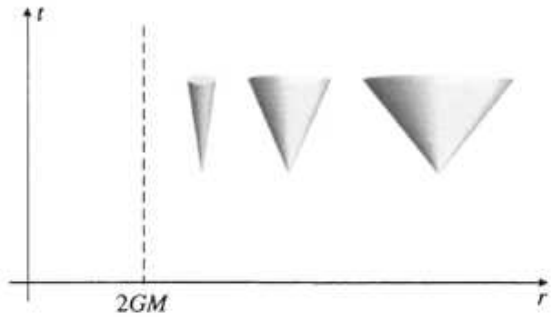


FIGURE 5.7 In Schwarzschild coordinates, light cones appear to close up as we approach $r = 2GM$.

the observer cross $r = 2GM$, we would just see them move more and more slowly (and become redder and redder, as if embarrassed to have done something as stupid as diving into a black hole).

The fact that we never see the infalling observer reach $r = 2GM$ is a meaningful statement, but the fact that their trajectory in the t-r plane never reaches there is not. It is highly dependent on our coordinate system, and we would like to ask a more coordinate-independent question (such as, "Does the observer reach this radius in a finite amount of their proper time?"). The best way to do this is to change coordinates to a system that is better behaved at $r = 2GM$. We now set out to find an appropriate set of such coordinates. There is no way to "derive" a coordinate transformation, of course, we just say what the new coordinates are and plug in the formulas. But we will develop these coordinates in several steps, in hopes of making the choices seem somewhat motivated.

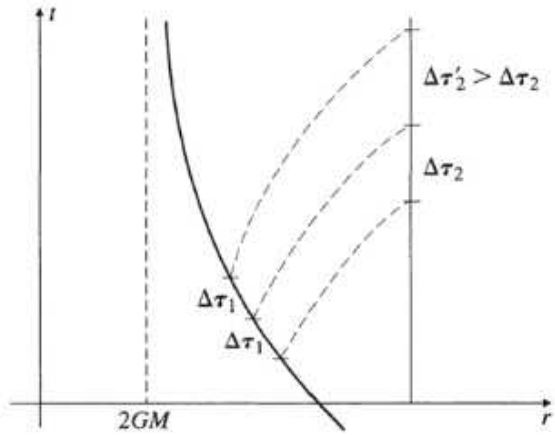


FIGURE 5.8 A beacon falling freely into a black hole emits signals at intervals of constant proper time $\Delta\tau_1$. An observer at fixed r receives the signals at successively longer time intervals $\Delta\tau_2$.

The problem with our current coordinates is that $dt/dr \to \infty$ along radial null geodesics that approach $r = 2GM$; progress in the r direction becomes slower and slower with respect to the coordinate time t. We can try to fix this problem by replacing t with a coordinate that moves more slowly along null geodesics. First notice that we can explicitly solve the condition (5.106) characterizing radial null curves to obtain

$$t = \pm r^* + \text{ constant}, \tag{5.107}$$

where the **tortoise coordinate** r^* is defined by

$$r^* = r + 2GM \ln\left(\frac{r}{2GM} - 1\right). \tag{5.108}$$

(The tortoise coordinate is only sensibly related to r when $r \geq 2GM$, but beyond there our coordinates aren't very good anyway.) In terms of the tortoise coordinate the Schwarzschild metric becomes

$$ds^2 = \left(1 - \frac{2GM}{r}\right)\left(-dt^2 + dr^{*2}\right) + r^2\, d\Omega^2, \tag{5.109}$$

where r is thought of as a function of r^*. This represents some progress, since the light cones now don't seem to close up, as shown in Figure 5.9; furthermore, none of the metric coefficients becomes infinite at $r = 2GM$ (although both g_{tt} and $g_{r^*r^*}$ become zero). The price we pay, however, is that the surface of interest at $r = 2GM$ has just been pushed to infinity.

Our next move is to define coordinates that are naturally adapted to the null geodesics. If we let

$$\begin{aligned} v &= t + r^* \\ u &= t - r^*, \end{aligned} \tag{5.110}$$

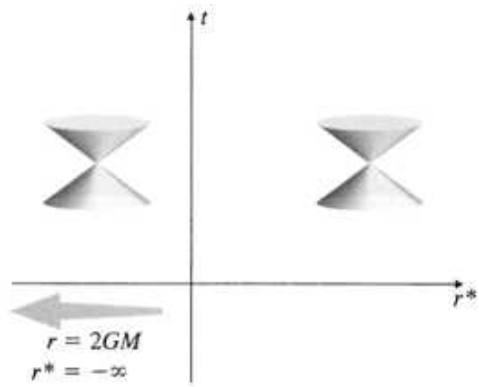


FIGURE 5.9 Schwarzschild light cones in tortoise coordinates, equation (5.109). Light cones remain nondegenerate, but the surface $r = 2GM$ has been pushed to infinity.

then infalling radial null geodesics are characterized by v = constant, while the outgoing ones satisfy u = constant. Now consider going back to the original radial coordinate r, but replacing the timelike coordinate t with the new coordinate v. These are known as **Eddington–Finkelstein coordinates.** In terms of these coordinates the metric is

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dv^2 + (dv\,dr + dr\,dv) + r^2 d\Omega^2. \tag{5.111}$$

Here we see our first sign of real progress. Even though the metric coefficient g_{vv} vanishes at $r = 2GM$, there is no real degeneracy; the determinant of the metric is

$$g = -r^4 \sin^2\theta, \tag{5.112}$$

which is perfectly regular at $r = 2GM$. Therefore the metric is invertible, and we see once and for all that $r = 2GM$ is simply a coordinate singularity in our original (t, r, θ, ϕ) system. In the Eddington–Finkelstein coordinates the condition for radial null curves is solved by

$$\frac{dv}{dr} = \begin{cases} 0, & \text{(infalling)} \\ 2\left(1 - \dfrac{2GM}{r}\right)^{-1}. & \text{(outgoing)} \end{cases} \tag{5.113}$$

We can therefore see what has happened: In this coordinate system the light cones remain well-behaved at $r = 2GM$, and this surface is at a finite coordinate value. There is no problem in tracing the paths of null or timelike particles past the surface. On the other hand, something interesting is certainly going on. Although the light cones don't close up, they do tilt over, such that for $r < 2GM$ all future-directed paths are in the direction of decreasing r, as shown in Figure 5.10.

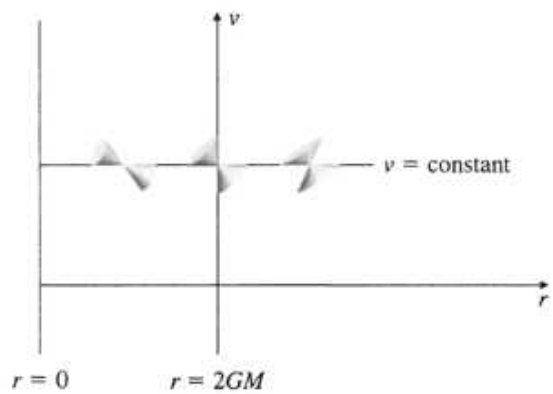


FIGURE 5.10 Schwarzschild light cones in the (v, r) coordinates of (5.111). In these coordinates we can follow future-directed timelike paths past $r = 2GM$.

The surface $r = 2GM$, while being locally perfectly regular, globally functions as a point of no return—once a test particle dips below it, it can never come back. We define an **event horizon** to be a surface past which particles can never escape to infinity; in Schwarzschild the event horizon is located at $r = 2GM$. (This is a rough definition; we will be somewhat more precise in the next chapter.) Despite being located at fixed radial coordinate, the event horizon is a null surface rather than a timelike one, so it is really the causal structure of spacetime itself that makes it impossible to cross the horizon in an outward-going direction. Since nothing can escape the event horizon, it is impossible for us to see inside—thus the name **black hole**. A black hole is simply a region of spacetime separated from infinity by an event horizon. The notion of an event horizon is a global one; the location of the horizon is a statement about the spacetime as a whole, not something you could determine just by knowing the geometry at that location. This will continue to be true in more general spacetimes.

We should mention a couple of features of black holes that sometimes get confused in the popular imagination. First, the external geometry of a black hole is the same Schwarzschild solution that we would have outside a star or planet. In particular, a black hole does not suck in everything around it any more than the Sun does; a particle well outside $r = 2GM$ behaves in exactly the same way regardless of whether the gravitating source is a black hole or not. Second, there is a misleading Newtonian analogy for black holes. The Newtonian escape velocity of a particle at distance r from a gravitating body of mass M is

$$v_{\rm esc} = \sqrt{\frac{2GM}{r}}. \tag{5.114}$$

If we naively ask where the Newtonian escape velocity equals the velocity of light, we find exactly $r = 2GM$. Despite the fact that the speed of light plays no fundamental role in Newtonian theory, it might seem provocative that light, thought of as inertial particles moving at a velocity c, is seemingly not able to escape from a body with mass M and radius less than $2GM$. But there is a profound difference between this case and what we see in GR. The escape velocity is the velocity that a particle would initially need to have in order to escape from a gravitating source on a free trajectory. But nothing stops us from considering accelerated trajectories; for example, one could imagine an acceleration chosen such that the particle moved steadily away from the massive body at some constant velocity. Therefore, a purported Newtonian black hole would not have the crucial property that *nothing* can escape; whereas in GR, arbitrary timelike paths must stay inside their light cones, and hence never escape the event horizon.

5.7 ■ THE MAXIMALLY EXTENDED SCHWARZSCHILD SOLUTION

Let's review what we have done. Acting under the suspicion that our coordinates may not have been good for the entire manifold, we have changed from our original coordinate t to the new one v, which has the nice property that if we decrease

r along a radial null curve v = constant, we go right through the event horizon without any problems. Indeed, a local observer actually making the trip would not necessarily know when the event horizon had been crossed—the local geometry is no different from anywhere else. We therefore conclude that our suspicion was correct and our initial coordinate system didn't do a good job of covering the entire manifold. The region $r \leq 2GM$ should certainly be included in our spacetime, since physical particles can easily reach there and pass through. However, there is no guarantee that we are finished; perhaps we can extend our manifold in other directions.

In fact there are other directions. In the (v, r) coordinate system we can cross the event horizon on future-directed paths, but not on past-directed ones. This seems unreasonable, since we started with a time-independent solution. But we could have chosen u instead of v, in which case the metric would have been

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) du^2 - (du\,dr + dr\,du) + r^2\, d\Omega^2. \tag{5.115}$$

Now we can once again pass through the event horizon, but this time only along past-directed curves, as shown in Figure 5.11.

This is perhaps a surprise: we can consistently follow either future-directed or past-directed curves through $r = 2GM$, but we arrive at different places. It was actually to be expected, since from the definitions (5.110), if we keep v constant and decrease r we must have $t \to +\infty$, while if we keep u constant and decrease r we must have $t \to -\infty$. (The tortoise coordinate r^* goes to $-\infty$ as $r \to 2GM$.) So we have extended spacetime in two different directions, one to the future and one to the past.

The next step would be to follow spacelike geodesics to see if we would uncover still more regions. The answer is yes, we would reach yet another piece of the spacetime, but let's shortcut the process by defining coordinates that are good all over. A first guess might be to use both u and v at once (in place of t and r),

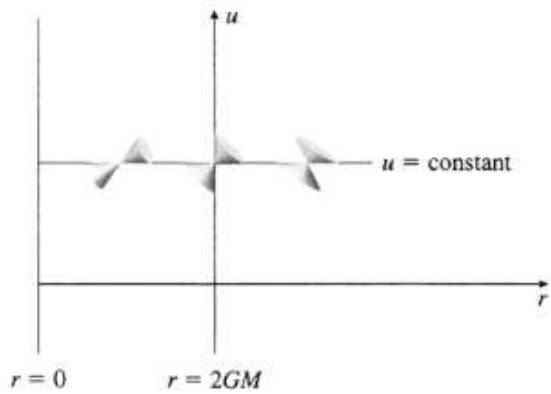


FIGURE 5.11 Schwarzschild light cones in the (u, r) coordinates of (5.115). In these coordinates we can follow past-directed timelike paths past $r = 2GM$.

which leads to

$$ds^2 = -\frac{1}{2}\left(1 - \frac{2GM}{r}\right)(\mathrm{d}v\mathrm{d}u + \mathrm{d}u\,\mathrm{d}v) + r^2\,d\Omega^2, \tag{5.116}$$

with r defined implicitly in terms of v and u by

$$\frac{1}{2}(v-u) = r + 2GM\ln\left(\frac{r}{2GM} - 1\right). \tag{5.117}$$

We have actually reintroduced the degeneracy with which we started out; in these coordinates $r = 2GM$ is "infinitely far away" (at either $v = -\infty$ or $u = +\infty$). The thing to do is to change to coordinates that pull these points into finite coordinate values; a good choice is

$$\begin{aligned} v' &= e^{v/4GM} \\ u' &= -e^{-u/4GM}, \end{aligned} \tag{5.118}$$

which in terms of our original (t, r) system is

$$\begin{aligned} v' &= \left(\frac{r}{2GM} - 1\right)^{1/2} e^{(r+t)/4GM} \\ u' &= -\left(\frac{r}{2GM} - 1\right)^{1/2} e^{(r-t)/4GM}. \end{aligned} \tag{5.119}$$

In the (v', u', θ, ϕ) system the Schwarzschild metric is

$$ds^2 = -\frac{16G^3M^3}{r}e^{-r/2GM}(\mathrm{d}v'\mathrm{d}u' + \mathrm{d}u'\mathrm{d}v') + r^2\,d\Omega^2. \tag{5.120}$$

Finally the nonsingular nature of $r = 2GM$ becomes completely manifest; in this form none of the metric coefficients behaves in any special way at the event horizon.

Both v' and u' are null coordinates, in the sense that their partial derivatives $\partial/\partial v'$ and $\partial/\partial u'$ are null vectors. There is nothing wrong with this, since the collection of four partial derivative vectors (two null and two spacelike) in this system serve as a perfectly good basis for the tangent space. Nevertheless, we are somewhat more comfortable working in a system where one coordinate is timelike and the rest are spacelike. We therefore define

$$T = \frac{1}{2}(v' + u') = \left(\frac{r}{2GM} - 1\right)^{1/2} e^{r/4GM}\sinh\left(\frac{t}{4GM}\right) \tag{5.121}$$

and

$$R = \frac{1}{2}(v' - u') = \left(\frac{r}{2GM} - 1\right)^{1/2} e^{r/4GM}\cosh\left(\frac{t}{4GM}\right), \tag{5.122}$$

in terms of which the metric becomes

$$\boxed{ds^2 = \frac{32G^3M^3}{r} e^{-r/2GM}(-\mathrm{d}T^2 + \mathrm{d}R^2) + r^2\, d\Omega^2,} \tag{5.123}$$

where r is defined implicitly from

$$T^2 - R^2 = \left(1 - \frac{r}{2GM}\right) e^{r/2GM}. \tag{5.124}$$

The coordinates (T, R, θ, ϕ) are known as **Kruskal coordinates**, or sometimes Kruskal–Szekeres coordinates.

The Kruskal coordinates have a number of miraculous properties. Like the (t, r^*) coordinates, the radial null curves look like they do in flat space:

$$T = \pm R + \text{constant}. \tag{5.125}$$

Unlike the (t, r^*) coordinates, however, the event horizon $r = 2GM$ is not infinitely far away; in fact it is defined by

$$T = \pm R, \tag{5.126}$$

consistent with it being a null surface. More generally, we can consider the surfaces $r = \text{constant}$. From (5.124) these satisfy

$$T^2 - R^2 = \text{constant}. \tag{5.127}$$

Thus, they appear as hyperbolae in the R-T plane. Furthermore, the surfaces of constant t are given by

$$\frac{T}{R} = \tanh\left(\frac{t}{4GM}\right), \tag{5.128}$$

which defines straight lines through the origin with slope $\tanh(t/4GM)$. Note that as $t \to \pm\infty$ (5.128) becomes the same as (5.126); therefore $t = \pm\infty$ represents the same surface as $r = 2GM$.

Our coordinates (T, R) should be allowed to range over every value they can take without hitting the real singularity at $r = 0$; the allowed region is therefore

$$\begin{aligned} -\infty \le R \le \infty \\ T^2 < R^2 + 1. \end{aligned} \tag{5.129}$$

From (5.121) and (5.122), T and R seem to become imaginary for $r < 2GM$, but this is an illusion; in that region the (r, t) coordinates are no good (specifically, $|t| > \infty$). We can now draw a spacetime diagram in the T-R plane (with θ and ϕ suppressed), known as a **Kruskal diagram**, shown in Figure 5.12. Each point on the diagram is a two-sphere. This diagram represents the maximal extension

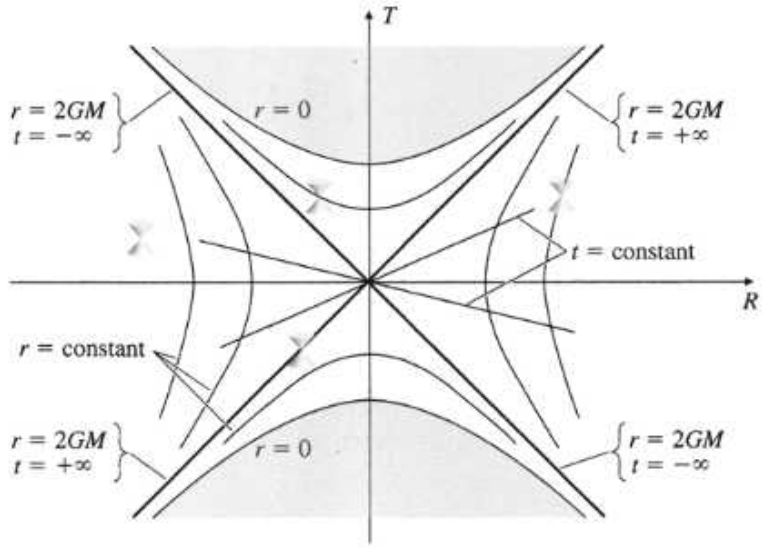


FIGURE 5.12 The Kruskal diagram—the Schwarzschild solution in Kruskal coordinates, where all light cones are at $\pm 45°$.

of the Schwarzschild geometry; the coordinates cover what we should think of as the entire manifold described by this solution.

The original Schwarzschild coordinates (t, r) were good for $r > 2GM$, which is only a part of the manifold portrayed on the Kruskal diagram. It is convenient to divide the diagram into four regions, as shown in Figure 5.13. Region I corresponds to $r > 2GM$, the patch in which our original coordinates were well-defined. By following future-directed null rays we reach region II, and by following past-directed null rays we reach region III. If we had explored spacelike geodesics, we would have been led to region IV. The definitions (5.121) and (5.122), which relate (T, R) to (t, r), are really only good in region I; in the other regions it is necessary to introduce appropriate minus signs to prevent the coordinates from becoming imaginary.

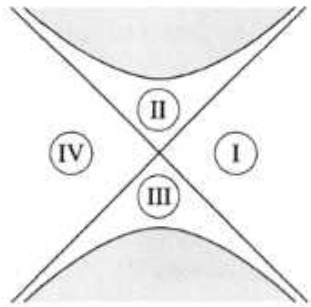


FIGURE 5.13 Regions of the Kruskal diagram.

Having extended the Schwarzschild geometry as far as it will go, we have described a remarkable spacetime. Region II, of course, is what we think of as the black hole. Once anything travels from region I into II, it can never return. In fact, every future-directed path in region II ends up hitting the singularity at $r = 0$; once you enter the event horizon, you are utterly doomed. This is worth stressing; not only can you not escape back to region I, you cannot even stop yourself from moving in the direction of decreasing r, since this is simply the timelike direction. This could have been seen in our original coordinate system; for $r < 2GM$, t becomes spacelike and r becomes timelike. Thus you can no more stop moving toward the singularity than you can stop getting older. Since proper time is maximized along a geodesic, you will live the longest if you don't struggle, but just relax as you approach the singularity. Not that you will have long to relax, nor will the voyage be very relaxing; as you approach the singularity the tidal forces become infinite. As you fall toward the singularity your feet and head will be pulled apart from each other, while your torso is squeezed to infinitesimal thinness. The grisly demise of an astrophysicist falling into a black hole is detailed in Misner, Thorne, and Wheeler (1973), Section 32.6. Note that they use orthonormal frames, as we discuss in Appendix J (not that it makes the trip any more enjoyable).

Regions III and IV might be somewhat unexpected. Region III is simply the time-reverse of region II, a part of spacetime from which things can escape to us, while we can never get there. It can be thought of as a **white hole**. There is a singularity in the past, out of which the universe appears to spring. The boundary of region III is the past event horizon, while the boundary of region II is the future event horizon. Region IV, meanwhile, cannot be reached from our region I either forward or backward in time, nor can anybody from over there reach us. It is another asymptotically flat region of spacetime, a mirror image of ours. It can be thought of as being connected to region I by a wormhole (or Einstein–Rosen bridge), a neck-like configuration joining two distinct regions. Consider slicing up the Kruskal diagram into spacelike surfaces of constant T, as shown in Figure 5.14. Now we can draw pictures of each slice, restoring one of the angular coordinates for clarity, as in Figure 5.15. In this way of slicing, the Schwarzschild geometry describes two asymptotically flat regions that reach toward each other, join together via a wormhole for a while, and then disconnect. But the wormhole closes up too quickly for any timelike observer to cross it from one region into the next.

As pleasing as the Kruskal diagram is, it is often even more useful to collapse the Schwarzschild solution into a finite region by constructing its conformal diagram. The idea of a conformal diagram is discussed in Appendix H; it is a crucial tool for analyzing spacetimes in general relativity, and you are encouraged to review that discussion now. We will not go through the manipulations necessary to construct the conformal diagram of Schwarzschild in full detail, since they parallel the Minkowski case with considerable additional algebraic complexity. We would start with the null version of the Kruskal coordinates, in which the metric

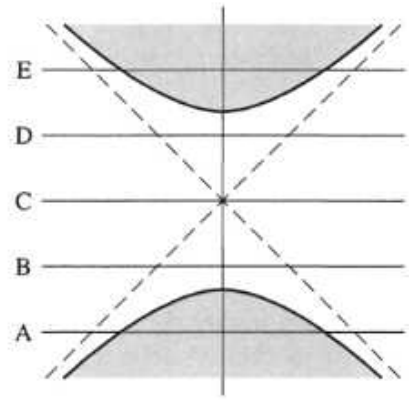


FIGURE 5.14 Spacelike slices in Kruskal coordinates.

takes the form

$$ds^2 = -\frac{16G^3M^3}{r}e^{-r/2GM}(\mathrm{d}v'\mathrm{d}u' + \mathrm{d}u'\mathrm{d}v') + r^2\,d\Omega^2, \tag{5.130}$$

where r is defined implicitly via

$$v'u' = -\left(\frac{r}{2GM} - 1\right)e^{r/2GM}. \tag{5.131}$$

Then essentially the same transformation used in the flat spacetime case suffices to bring infinity into finite coordinate values:

$$\begin{aligned} v'' &= \arctan\left(\frac{v'}{\sqrt{2GM}}\right) \\ u'' &= \arctan\left(\frac{u'}{\sqrt{2GM}}\right), \end{aligned} \tag{5.132}$$

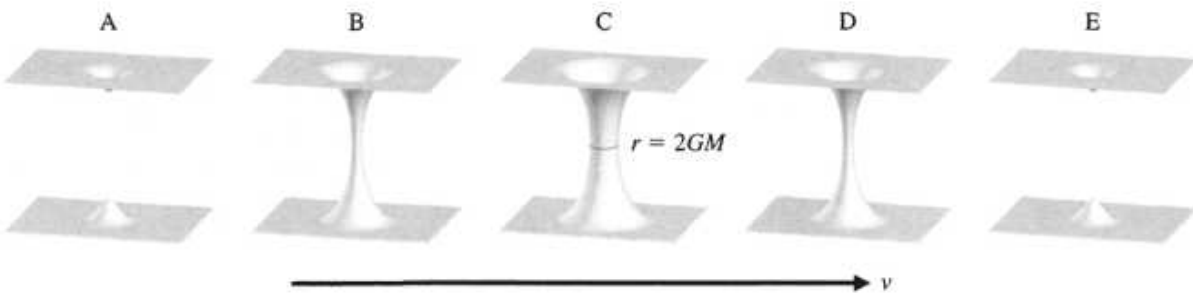


FIGURE 5.15 Geometry of the spacelike slices in Figure 5.14.

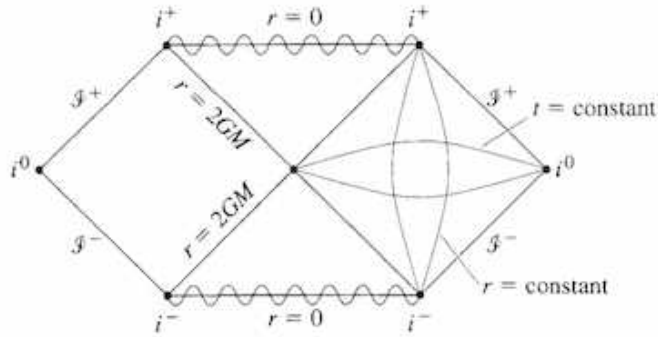


FIGURE 5.16 Conformal diagram for Schwarzschild spacetime.

with ranges

$$-\frac{\pi}{2} < v'' < +\frac{\pi}{2}$$
$$-\frac{\pi}{2} < u'' < +\frac{\pi}{2}$$
$$-\frac{\pi}{2} < v'' + u'' < \frac{\pi}{2}.$$

The (v'', u'') part of the metric (that is, at constant angular coordinates) is now conformally related to Minkowski space. In the new coordinates the singularities at $r = 0$ are straight lines that stretch from timelike infinity in one asymptotic region to timelike infinity in the other.

The conformal diagram for the maximally extended Schwarzschild solution thus looks like Figure 5.16. The only real subtlety about this diagram is the necessity to understand that i^+ and i^- (future and past infinity) are distinct from $r = 0$—there are plenty of timelike paths that do not hit the singularity. As in the Kruskal diagram, light cones in the conformal diagram are at 45°; the major difference is that the entire spacetime is represented in a finite region. Notice also that the structure of conformal infinity is just like that of Minkowski space, consistent with the claim that Schwarzschild is asymptotically flat.

5.8 ■ STARS AND BLACK HOLES

The maximally extended Schwarzschild solution we have just constructed tells a remarkable story, including not only the sought-after black hole, but also a white hole and an additional asymptotically flat region, connected to our universe by a wormhole. It would be premature, however, to imagine that such features are common in the real world. The Schwarzschild solution represents a highly idealized situation: not only spherically symmetric, but completely free of energy-momentum throughout spacetime. Birkhoff's theorem implies that any vacuum

region of a spherically symmetric spacetime will be described by *part of* the Schwarzschild metric, but the existence of matter somewhere in the universe may dramatically alter the global picture.

A static spherical object—let's call it a star for definiteness—with radius larger than $2GM$ will be Schwarzschild in the exterior, but there won't be any singularities or horizons, and the global structure will actually be very similar to Minkowski spacetime. Of course, real stars evolve, and it may happen that a star eventually collapses under its own gravitational pull, shrinking down to below $r = 2GM$ and further into a singularity, resulting in a black hole. There is no need for a white hole, however, because the past of such a spacetime looks nothing like that of the full Schwarzschild solution. A conformal diagram describing stellar collapse would look like Figure 5.17. The interior shaded region is nonvacuum, so is not described by Schwarzschild; in particular, there is no wormhole connecting to another universe. It is asymptotically Minkowskian, except for a future region giving rise to an event horizon. We see that a realistic black hole may share the singularity and future horizon with the maximally extended Schwarzschild solution, without any white hole, past horizon, or separate asymptotic region.

We believe that gravitational collapse of this kind is by no means a necessary endpoint of stellar evolution, but will occur under certain conditions. General relativity places rigorous limits on the kind of stars that can resist gravitational collapse; for any given sort of matter, enough mass will always lead to the collapse to a black hole. Furthermore, from astrophysical observations we have excellent evidence that black holes exist in our universe.

To understand gravitational collapse to a black hole, we should first understand static configurations describing the interiors of spherically symmetric stars. We won't delve into this subject in detail, only enough to get a feeling for the basic features of interior solutions. Consider the general static, spherically symmetric

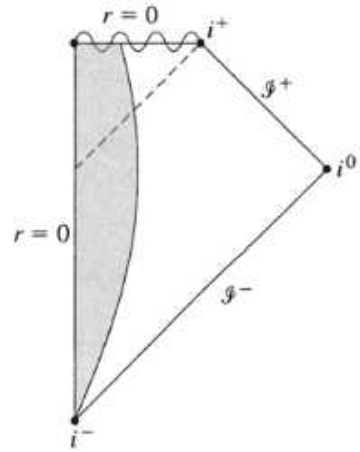


FIGURE 5.17 Conformal diagram for a black hole formed from a collapsing star. The shaded region contains matter, and will be described by an appropriate dynamical interior solution; the exterior region is Schwarzschild.

metric from (5.11):

$$ds^2 = -e^{2\alpha(r)}\,dt^2 + e^{2\beta(r)}\,dr^2 + r^2\,d\Omega^2. \tag{5.133}$$

We are now looking for nonvacuum solutions, so we turn to the full Einstein equation,

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{5.134}$$

The Einstein tensor follows from the Ricci tensor (5.14) and curvature scalar (5.15),

$$\begin{aligned}
G_{tt} &= \frac{1}{r^2}e^{2(\alpha-\beta)}\left(2r\partial_r\beta - 1 + e^{2\beta}\right)\\
G_{rr} &= \frac{1}{r^2}\left(2r\partial_r\alpha + 1 - e^{2\beta}\right)\\
G_{\theta\theta} &= r^2 e^{-2\beta}\left[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \frac{1}{r}(\partial_r\alpha - \partial_r\beta)\right]\\
G_{\phi\phi} &= \sin^2\theta\, G_{\theta\theta}.
\end{aligned} \tag{5.135}$$

We model the star itself as a perfect fluid, with energy-momentum tensor

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu}. \tag{5.136}$$

The energy density ρ and pressure p will be functions of r alone. Since we seek static solutions, we can take the four-velocity to be pointing in the timelike direction. Normalized to $U^\mu U_\mu = -1$, it becomes

$$U_\mu = (e^\alpha, 0, 0, 0), \tag{5.137}$$

so that the components of the energy-momentum tensor are

$$T_{\mu\nu} = \begin{pmatrix} e^{2\alpha}\rho & & & \\ & e^{2\beta}p & & \\ & & r^2 p & \\ & & & r^2(\sin^2\theta)p \end{pmatrix}. \tag{5.138}$$

We therefore have three independent components of Einstein's equation: the tt component,

$$\frac{1}{r^2}e^{-2\beta}\left(2r\partial_r\beta - 1 + e^{2\beta}\right) = 8\pi G\rho, \tag{5.139}$$

the rr component,

$$\frac{1}{r^2}e^{-2\beta}\left(2r\partial_r\alpha + 1 - e^{2\beta}\right) = 8\pi G p, \tag{5.140}$$

and the $\theta\theta$ component,

$$e^{-2\beta}\left[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \frac{1}{r}(\partial_r\alpha - \partial_r\beta)\right] = 8\pi G p. \tag{5.141}$$

The $\phi\phi$ equation is proportional to the $\theta\theta$ equation, so there is no need to consider it separately.

We notice that the tt equation (5.139) involves only β and ρ. It is convenient to replace $\beta(r)$ with a new function $m(r)$, given by

$$m(r) = \frac{1}{2G}(r - re^{-2\beta}), \tag{5.142}$$

or equivalently

$$e^{2\beta} = \left[1 - \frac{2Gm(r)}{r}\right]^{-1}, \tag{5.143}$$

so that

$$ds^2 = -e^{2\alpha(r)}\,\mathrm{d}t^2 + \left[1 - \frac{2Gm(r)}{r}\right]^{-1}\mathrm{d}r^2 + r^2\,d\Omega^2. \tag{5.144}$$

The metric component g_{rr} is an obvious generalization of the Schwarzschild case, but this will not be true for g_{tt}. The tt equation (5.139) becomes

$$\frac{dm}{dr} = 4\pi r^2\rho, \tag{5.145}$$

which can be integrated to obtain

$$m(r) = 4\pi\int_0^r \rho(r')r'^2dr'. \tag{5.146}$$

Let's imagine that our star extends to a radius R, after which we are in vacuum and described by Schwarzschild. In order that the metrics match at this radius, the Schwarzschild mass M must be given by

$$M = m(R) = 4\pi\int_0^R \rho(r)r^2dr. \tag{5.147}$$

It looks like $m(r)$ is simply the integral of the energy density over the stellar interior, and can be interpreted as the mass within a radius r.

There is one subtlety with interpreting $m(r)$ as the integrated energy density; in a proper spatial integral, the volume element should be

$$\sqrt{\gamma}\,d^3x = e^{\beta}r^2\sin\theta\,drd\theta d\phi, \tag{5.148}$$

where

$$\gamma_{ij}\,\mathrm{d}x^i\mathrm{d}x^j = e^{2\beta}\,\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2 \tag{5.149}$$

is the spatial metric. The true integrated energy density is therefore

$$\begin{aligned}\bar{M} &= 4\pi \int_0^R \rho(r) r^2 e^{\beta(r)}\, dr \\ &= 4\pi \int_0^R \frac{\rho(r) r^2}{\left[1 - \frac{2Gm(r)}{r}\right]^{1/2}}\, dr. \end{aligned} \tag{5.150}$$

The difference, of course, arises because there is a binding energy due to the mutual gravitational attraction of the fluid elements in the star, which is given by

$$E_{\rm B} = \bar{M} - M > 0. \tag{5.151}$$

The binding energy is the amount of energy that would be required to disperse the matter in the star to infinity. It is not always a well-defined notion in general relativity, but makes sense for spherical stars.

In terms of $m(r)$, the rr equation (5.140) can be written

$$\frac{d\alpha}{dr} = \frac{Gm(r) + 4\pi G r^3 p}{r[r - 2Gm(r)]}. \tag{5.152}$$

It is convenient not to use the $\theta\theta$ equation directly, but instead appeal to energy-momentum conservation, $\nabla_\mu T^{\mu\nu} = 0$. For our metric (5.144), it is straightforward to derive that $\nu = r$ is the only nontrivial component, and it gives

$$(\rho + p)\frac{d\alpha}{dr} = -\frac{dp}{dr}. \tag{5.153}$$

Combining this with (5.152) allows us to eliminate $\alpha(r)$ to obtain

$$\frac{dp}{dr} = -\frac{(\rho + p)[Gm(r) + 4\pi G r^3 p]}{r[r - 2Gm(r)]}. \tag{5.154}$$

This is the **Tolman–Oppenheimer–Volkoff equation**, or simply the equation of hydrostatic equilibrium. Since $m(r)$ is related to $\rho(r)$ via (5.146), this equation relates $p(r)$ to $\rho(r)$. To get a closed system of equations, we need one more relation: the equation of state. In general this will give the pressure in terms of the energy density and specific entropy, $p = p(\rho, S)$. Often we care about situations in which the entropy is very small, and can be neglected; the equation of state then takes the form

$$p = p(\rho). \tag{5.155}$$

Astrophysical systems often obey a polytropic equation of state, $p = K\rho^\gamma$ for some constants K and γ.

A simple and semi-realistic model of a star comes from assuming that the fluid is incompressible: the density is a constant ρ_* out to the surface of the star, after

which it vanishes,

$$\rho(r) = \begin{cases} \rho_*, & r < R \\ 0, & r > R. \end{cases} \tag{5.156}$$

Specifying $\rho(r)$ explicitly takes the place of an equation of state, since $p(r)$ can be determined from hydrostatic equilibrium. It is then straightforward to integrate (5.146) to get

$$m(r) = \begin{cases} \frac{4}{3}\pi r^3 \rho_*, & r < R \\ \frac{4}{3}\pi R^3 \rho_* = M, & r > R. \end{cases} \tag{5.157}$$

Integrating the equation of hydrostatic equilibrium yields

$$p(r) = \rho_* \left[\frac{R\sqrt{R-2GM} - \sqrt{R^3 - 2GMr^2}}{\sqrt{R^3 - 2GMr^2} - 3R\sqrt{R-2GM}} \right]. \tag{5.158}$$

Finally we can get the metric component $g_{tt} = -e^{2\alpha(r)}$ from (5.152); we find that

$$e^{\alpha(r)} = \frac{3}{2}\left(1 - \frac{2GM}{R}\right)^{1/2} - \frac{1}{2}\left(1 - \frac{2GMr^2}{R^3}\right)^{1/2}, \quad r < R. \tag{5.159}$$

The pressure increases near the core of the star, as one would expect. Indeed, for a star of fixed radius R, the central pressure $p(0)$ will need to be greater than infinity if the mass exceeds

$$M_{\max} = \frac{4}{9G}R. \tag{5.160}$$

Thus, if we try to squeeze a greater mass than this inside a radius R, general relativity admits no static solutions; a star that shrinks to such a size must inevitably keep shrinking, eventually forming a black hole. We derived this result from the rather strong assumption that the density is constant, but it continues to hold when that assumption considerably weakened; **Buchdahl's theorem** states that any reasonable static, spherically symmetric interior solution has $M < 4R/9G$. Although a careful proof requires more work, this result makes sense; if we imagine that there is some maximum sustainable density in nature, the most massive object we could in principle make would have that density everywhere, which is the specific case we considered.

Of course, this still doesn't mean that realistic astrophysical objects will always ultimately collapse to black holes. An ordinary planet, supported by material pressures, will persist essentially forever (apart from some fantastically unlikely quantum tunneling from a planet to something very different, or the possibility of eventual proton decay). But massive stars are a different story. The pressure supporting a star comes from the heat produced by fusion of light nuclei into heavier ones. When the nuclear fuel is used up, the temperature declines and the

star begins to shrink under the influence of gravity. The collapse may eventually be halted by Fermi degeneracy pressure: Electrons are pushed so close together that they resist further compression simply on the basis of the Pauli exclusion principle (no two fermions can be in the same state). A stellar remnant supported by electron degeneracy pressure is called a **white dwarf**; a typical white dwarf is comparable in size to the Earth. Lower-mass particles become degenerate at lower number densities than high-mass particles, so nucleons do not contribute appreciably to the pressure in a white dwarf. White dwarfs are the end state for most stars, and are extremely common throughout the universe.

If the total mass is sufficiently high, however, the star will reach the **Chandrasekhar limit**, where even the electron degeneracy pressure is not enough to resist the pull of gravity. Calculations put the Chandrasekhar limit at about $1.4\ M_\odot$, where $M_\odot = 2 \times 10^{33}$ g is the mass of the Sun. When it is reached, the star is forced to collapse to an even smaller radius. At this point electrons combine with protons to make neutrons and neutrinos (inverse beta decay), and the neutrinos simply fly away. The result is a **neutron star**, with a typical radius of about 10 km. Neutron stars have a low total luminosity, but often are rapidly spinning and possess strong magnetic fields. This combination gives rise to **pulsars**, which accelerate particles in jets emanating from the magnetic poles, appearing to rapidly flash as the neutron star spins. Pulsars were discovered by Bell in 1967; after a brief speculation that they might represent signals from an extraterrestrial civilization, the more prosaic astrophysical explanation was settled on.

Since the conditions at the center of a neutron star are very different from those on Earth, we do not have a perfect understanding of the equation of state. Nevertheless, we believe that a sufficiently massive neutron star will itself be unable to resist the pull of gravity, and will continue to collapse; current estimates of the maximum possible neutron-star mass are around 3–4 $M_\odot$, the **Oppenheimer-Volkoff limit**. Since a fluid of neutrons is the densest material we know about (apart from some very speculative suggestions), it is believed that the outcome of such a collapse is a black hole.

How would we know if there were a black hole? The fundamental obstacle to direct detection is, of course, blackness: a black hole will not itself give off any radiation (neglecting Hawking radiation, which is a very small effect to be discussed in Chapter 9). But black holes will feature extremely strong gravitational fields, so we can hope to detect them indirectly by observing matter being influenced by these fields. As matter falls into a black hole, it will heat up and emit X-rays, which we can detect with satellite observatories. A large number of black-hole candidates have been detected by this method, and the case for real black holes in our universe is extremely strong.[1] The large majority of candidates fall into one of two classes. There are black holes with masses of order a solar mass or somewhat higher; these are thought to be the endpoints of evolution for very massive stars. The other category describes supermassive black holes, be-

[1] For a review on astrophysical evidence for black holes, see A. Celotti, J.C. Miller, and D.W. Sciama (1999), *Class. Quant. Grav.* **16**, A3; `http://arxiv.org/abs/astro-ph/9912186`.

tween 10^6 and 10^9 solar masses. These are found at the centers of galaxies, and are thought to be the engines that powered quasars in the early era of galaxy formation. Our own Milky Way galaxy contains an object (Sgr A*) that is believed to be a black hole of at least $2 \times 10^6 M_\odot$. The precise history of the formation of these supermassive holes is not well understood. Other possibilities include very small primordial black holes produced in the very early universe, and so-called "middleweight" black holes of order a thousand solar masses.

As matter falls into a black hole, it tends to settle into a rotating accretion disk, and both energy and angular momentum are gradually fed into the hole. As a result, the black holes we expect to see in astrophysical situations should be spinning, and indeed observations are consistent with very high spin rates for observed black holes. In this chapter we have excluded the possibility of black hole spin by focusing on the spherically symmetric Schwarzschild solution; in the next chapter we turn to more general types of black holes.

5.9 ■ EXERCISES

1. A space monkey is happily orbiting a Schwarzschild black hole in a circular geodesic orbit. An evil baboon, far from the black hole, tries to send the monkey to its death inside the black hole by dropping a carefully timed coconut radially toward the black hole, knowing that the monkey can't resist catching the falling coconut. Given the monkey's mass and initial orbital radius and the mass of the coconut, explain how you would go about solving the problem (but do not do the calculation). What are the possible fates for our intrepid space monkey?

2. Consider a perfect fluid in a static, circularly symmetric $(2+1)$-dimensional spacetime, equivalently, a cylindrical configuration in $(3+1)$ dimensions with perfect rotational symmetry.

   (a) Derive the analogue of the Tolman–Oppenhiemer–Volkov (TOV) equation for $(2+1)$ dimensions.

   (b) Show that the vacuum solution can be written as

   $$ds^2 = -dt^2 + \frac{1}{1-8GM} dr^2 + r^2 d\theta^2$$

   Here M is a constant.

   (c) Show that another way to write the same solution is

   $$ds^2 = -d\tau^2 + d\xi^2 + \xi^2 d\phi^2$$

   where $\phi \in [0, 2\pi(1-8GM)^{1/2}]$.

   (d) Solve the $(2+1)$ TOV equation for a constant density star. Find $p(r)$ and solve for the metric.

   (e) Solve the $(2+1)$ TOV equation for a star with equation of state $p = \kappa\rho^{3/2}$. Find $p(r)$ and solve for the metric.

   (f) Find the mass $M(R) = \int_0^{2\pi}\int_0^R \rho dr d\theta$ and the proper mass $\tilde{M}(R) = \int_0^{2\pi}\int_0^R \rho\sqrt{-g}\, dr\, d\theta$ for the solutions in parts (d) and (e).

3. Consider a particle (not necessarily on a geodesic) that has fallen inside the event horizon, $r < 2GM$. Use the ordinary Schwarzschild coordinates (t, r, θ, ϕ). Show that the radial coordinate must decrease at a minimum rate given by

$$\left|\frac{dr}{d\tau}\right| \geq \sqrt{\frac{2GM}{r} - 1}.$$

Calculate the maximum lifetime for a particle along a trajectory from $r = 2GM$ to $r = 0$. Express this in seconds for a black hole with mass measured in solar masses. Show that this maximum proper time is achieved by falling freely with $E \to 0$.

4. Consider Einstein's equations in vacuum, but with a cosmological constant, $G_{\mu\nu} + \Lambda g_{\mu\nu} = 0$.

   (a) Solve for the most general spherically symmetric metric, in coordinates (t, r) that reduce to the ordinary Schwarzschild coordinates when $\Lambda = 0$.

   (b) Write down the equation of motion for radial geodesics in terms of an effective potential, as in (5.66). Sketch the effective potential for massive particles.

5. Consider a comoving observer sitting at constant spatial coordinates (r_*, θ_*, ϕ_*), around a Schwarzschild black hole of mass M. The observer drops a beacon into the black hole (straight down, along a radial trajectory). The beacon emits radiation at a constant wavelength λ_{em} (in the beacon rest frame).

   (a) Calculate the coordinate speed dr/dt of the beacon, as a function of r.

   (b) Calculate the proper speed of the beacon. That is, imagine there is a comoving observer at fixed r, with a locally inertial coordinate system set up as the beacon passes by, and calculate the speed as measured by the comoving observer. What is it at $r = 2GM$?

   (c) Calculate the wavelength λ_{obs}, measured by the observer at r_*, as a function of the radius r_{em} at which the radiation was emitted.

   (d) Calculate the time t_{obs} at which a beam emitted by the beacon at radius r_{em} will be observed at r_*.

   (e) Show that at late times, the redshift grows exponentially: $\lambda_{\text{obs}}/\lambda_{\text{em}} \propto e^{t_{\text{obs}}/T}$. Give an expression for the time constant T in terms of the black hole mass M.

CHAPTER

6 More General Black Holes

6.1 ■ THE BLACK HOLE ZOO

Birkhoff's theorem ensures that the Schwarzschild metric is the only spherically symmetric vacuum solution to general relativity. This shouldn't be too surprising, as it is reminiscent of the situation in electromagnetism, where the only spherically symmetric field configuration in a region free of charges will be a Coulomb field. Moving beyond spherical symmetry, there is an unlimited variety of possible gravitational fields. For a planet like the Earth, for example, the external field will depend on the density and profile of all the various mountain ranges and valleys on the surface. We could imagine decomposing the metric into multipole moments, and an infinite number of coefficients would have to be specified to describe the field exactly.

It might therefore come as something of a surprise that black holes do not share this property. Only a small number of stationary black-hole solutions exist, described by a small number of parameters. The specific set of parameters will depend on what matter fields we include in our theory; if electromagnetism is the only long-range nongravitational field, we have a **no-hair theorem**:

> Stationary, asymptotically flat black hole solutions to general relativity coupled to electromagnetism that are nonsingular outside the event horizon are fully characterized by the parameters of mass, electric and magnetic charge, and angular momentum.

Stationary solutions are of special interest because we expect them to be the end states of gravitational collapse. The alternative might be some sort of oscillating configuration, but oscillations will ultimately be damped as energy is lost through the emission of gravitational radiation; in fact, typical evolutions will evolve quite rapidly to a stationary configuration.

We speak of "a" no-hair theorem, rather than "the" no-hair theorem, because the result depends not only on general relativity, but also on the matter content of our theory. In the Standard Model of particle physics, electromagnetism is the only long-range field, and the above theorem applies; but for different kinds of fields there might be other sorts of hair.[1] Examples have even been found of static (nonrotating) black holes that are axisymmetric but not completely spherically

[1] For a discussion see M. Heusler, "Stationary Black Holes: Uniqueness and Beyond," Living Rev. Relativity 1, (1998), 6; http://www.livingreviews.org/Articles/Volume1/1998-6heusler/.

symmetric. The central point, however, remains unaltered: black hole solutions are characterized by a very small number of parameters, rather than the potentially infinite set of parameters characterizing, say, a planet.

As we will discuss at the end of this chapter and again in Chapter 9, the no-hair property leads to a puzzling situation. In most physical theories, we hope to have a well-defined initial value problem, so that information about a state at any one moment of time can be used to predict (or retrodict) the state at any other moment of time. As a consequence, any two states that are connected by a solution to the equations of motion should require the same amount of information to be specified. But in GR, it seems, we can take a very complicated collection of matter, collapse it into a black hole, and end up with a configuration described completely by mass, charge, and spin. In classical GR this might not bother us so much, since the information can be thought of as hidden behind the event horizon rather than truly being lost. But when quantum field theory is taken into account, we find that black holes evaporate and eventually disappear, and the information seems to be truly lost. Conceivably, the outgoing Hawking radiation responsible for the evaporation somehow encodes information about what state was originally used to make a black hole, but how that could happen is completely unclear. Understanding this "information loss paradox" is considered by many to be a crucial step in building a sensible theory of quantum gravity.

In this chapter, however, we will stick to considerations of classical GR. We begin with some general discussion of black hole properties, especially those of event horizons and Killing horizons. This subject can be subtle and technical, and our philosophy here will be to try to convey the main ideas without being rigorous about definitions or proofs of theorems. We then discuss the specific solutions corresponding to charged (Reissner–Nordström) and spinning (Kerr) black holes; consistent with our approach, we will not carefully go through the coordinate redefinitions necessary to construct the maximally extended spacetimes, but instead simply draw the associated conformal diagrams. The reader interested in further details should consult the review article by Townsend,[2] or the books by Hawking and Ellis (1973) and Wald (1984), all of which we draw on heavily in this chapter.

6.2 ■ EVENT HORIZONS

Black holes are characterized by the fact that you can enter them, but never exit. Thus, their most important feature is actually not the singularity at the center, but the event horizon at the boundary. An event horizon is a hypersurface separating those spacetime points that are connected to infinity by a timelike path from those that are not. To understand what this means in practice, we should think a little more carefully about what we mean by "infinity." In general relativity, the global structure of spacetime can take many different forms, with correspondingly different notions of infinity. But to think about black holes in the real universe, we

[2] P.K. Townsend, "Black Holes: Lecture Notes," `http://arxiv.org/abs/gr-qc/9707012`.

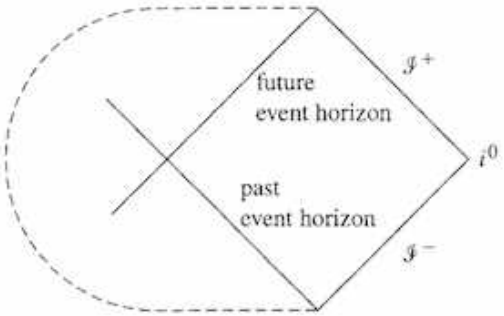


FIGURE 6.1 An asymptotically flat spacetime is one for which infinity in a conformal diagram matches that of Minkowski spacetime, with future null infinity $\mathscr{I}^+$, spacelike infinity i^0, and past null infinity $\mathscr{I}^-$. The future event horizon is the boundary of the past of $\mathscr{I}^+$. The dashed region represents the rest of the spacetime, which may take a number of different forms in different examples.

aren't actually concerned with what happens infinitely far away; we use infinity as a proxy for "well outside the black hole," and imagine that spacetime sufficiently far away from the hole can be approximated by Minkowski space.

As mentioned in the Chapter 5, a spacetime that looks Minkowskian at infinity is referred to as asymptotically flat. The meaning of this concept is made clear in a conformal diagram such as in Figure 6.1. From our discussion in Appendix H of the conformal diagram for Minkowski, we know that conformal infinity comes in five pieces: future and past timelike infinity $i^{\pm}$, future and past null infinity $\mathscr{I}^{\pm}$, and spatial infinity i^0. An asymptotically flat spacetime (or region of spacetime) is one for which $\mathscr{I}^{\pm}$ and i^0 have the same structure as for Minkowski; timelike infinity is not necessary. Such spacetimes will have the general form shown in Figure 6.1.

With this picture, it is clear how we should think of the future event horizon: it is the surface beyond which timelike curves cannot escape to infinity. Recalling that the causal past J^- of a region is the set of all points we can reach from that region by moving along past-directed timelike paths, the event horizon can be equivalently defined as the boundary of $J^-(\mathscr{I}^+)$, the causal past of future null infinity. (The event horizon is really the boundary of the *closure* of this set, but we're not being rigorous.) Analogous definitions hold for the past horizon. As we have seen in the case of maximally extended Schwarzschild, there may be more than one asymptotically flat region in a spacetime, and correspondingly more than one event horizon.

From the definition, it is clear that the event horizon is a null hypersurface. Properties of null hypersurfaces are discussed in Appendix D; here we can recall the major features. A hypersurface Σ can be defined by $f(x) =$ constant for some function $f(x)$. The gradient $\partial_\mu f$ is normal to Σ; if the normal vector is null, the hypersurface is said to be null, and the normal vector is also tangent to Σ. Null hypersurfaces can be thought of as a collection of null geodesics $x^\mu(\lambda)$, called the generators of the hypersurface. The tangent vectors ξ^μ to these geodesics are

proportional to the normal vectors,

$$\xi^\mu = \frac{dx^\mu}{d\lambda} = h(x) g^{\mu\nu} \partial_\nu f, \tag{6.1}$$

and therefore also serve as normal vectors to the hypersurface. We may choose the function $h(x)$ so that the geodesics are affinely parameterized, so the tangent vectors will obey

$$\xi_\mu \xi^\mu = 0, \quad \xi^\mu \nabla_\mu \xi^\nu = 0. \tag{6.2}$$

For future event horizons, the generators may end in the past (for example, when a black hole is formed by stellar collapse) but will always continue indefinitely into the future (and similarly with future and past interchanged).

Because the event horizon is a global concept, it might be difficult to actually locate one when you are handed a metric in an arbitrary set of coordinates. Fortunately, in this chapter we will be concerned with quite special metrics—stationary, asymptotically flat, and containing event horizons with spherical topology. In such spacetimes, there are convenient coordinate systems in which there is a simple way to identify the event horizon. For the Schwarzschild solution, the event horizon is a place where the light cones "tilt over" so that $r = 2GM$ is a null surface rather than a timelike surface, as $r =$ constant would be for large r. Light-cone tilting is clearly a coordinate-dependent notion (it doesn't happen, for example, in Kruskal coordinates), but the metrics of concern to us will allow for analogous constructions. A stationary metric has a Killing vector ∂_t that is asymptotically timelike, and we can adapt the metric components to be time-independent ($\partial_t g_{\mu\nu} = 0$). On hypersurfaces $t =$ constant, we can choose coordinates (r, θ, ϕ) in which the metric at infinity looks like Minkowski space in spherical polar coordinates. Hypersurfaces $r =$ constant will be timelike cylinders with topology $S^2 \times \mathbf{R}$ at $r \to \infty$. Now imagine we have chosen our coordinates cleverly, so that as we decrease r from infinity the $r =$ constant hypersurfaces remain timelike until some fixed $r = r_\mathrm{H}$, for which the surface is everywhere null. (In nonclever coordinates, $r =$ constant hypersurfaces will become null or spacelike for some values of θ and ϕ but remain timelike for others.) This will clearly represent an event horizon, since timelike paths crossing into the region $r < r_\mathrm{H}$ will never be able to escape back to infinity. Determining the point at which $r =$ constant hypersurfaces become null is easy; $\partial_\mu r$ is a one-form normal to such hypersurfaces, with norm

$$g^{\mu\nu}(\partial_\mu r)(\partial_\nu r) = g^{rr}. \tag{6.3}$$

We are looking for the place where the norm of our one-form vanishes; hence, in the coordinates we have described, the event horizon $r = r_\mathrm{H}$ will simply be the hypersurface at which g^{rr} switches from being positive to negative,

$$g^{rr}(r_\mathrm{H}) = 0. \tag{6.4}$$

This criterion clearly works for Schwarzschild, for which $g^{rr} = 1 - 2GM/r$. We will present the Reissner–Nordström and Kerr solutions in coordinates that are similarly adapted to the horizons.

The reason why we make such a big deal about event horizons is that they are nearly inevitable in general relativity. This conclusion is reached by concatenating two interesting results: Singularities are nearly inevitable, and singularities are hidden behind event horizons. Of course both results hold under appropriate sets of assumptions; it is not that hard to come up with spacetimes that have no singularities (Minkowski would be an example), nor is it even that hard to find singularities without horizons (as we will see below in our discussion of charged black holes). But we believe that "generic" solutions will have singularities hidden behind horizons.

The ubiquity of singularities is guaranteed by the **singularity theorems** of Hawking and Penrose. Before these theorems were proven, it was possible to hope that collapse to a Schwarzschild singularity was an artifact of spherical symmetry, and typical geometries would remain nonsingular (as happens, for example, in Newtonian gravity). But the Hawking–Penrose theorems demonstrate that once collapse reaches a certain point, evolution to a singularity is inevitable. The way we know there is a singularity is through geodesic incompleteness—there exists some geodesic that cannot be extended within the manifold, but nevertheless ends at a finite value of the affine parameter. The way we know collapse has reached a point of no return is the appearance of a **trapped surface**. To understand what a trapped surface is, first picture a two-sphere in Minkowski space, taken as a set of points some fixed radial distance from the origin, embedded in a constant-time slice. If we follow null rays emanating into spacetime from this spatial sphere, one set (pointed inward) will describe a shrinking set of spheres, while the other (pointed outward) will describe a growing set of spheres. But this would not be the case for a sphere of fixed radius $r < 2GM$ in the Schwarzschild geometry; inside the event horizon, both sets of null rays emanating from such a sphere would evolve to smaller values of r (since r is a timelike coordinate), and thus to smaller areas $4\pi r^2$. This is what is meant by a trapped surface: a compact, spacelike, two-dimensional submanifold with the property that outgoing future-directed light rays *converge* in both directions everywhere on the submanifold. (The formal definition of "converge" is that the expansion θ, as described in the discussion of geodesic congruences in Appendix F, is negative.)

With these definitions in hand, we can present an example of a singularity theorem.

> Let M be a manifold with a generic metric $g_{\mu\nu}$, satisfying Einstein's equation with the strong energy condition imposed. If there is a trapped surface in M, there must be either a closed timelike curve or a singularity (as manifested by an incomplete timelike or null geodesic).

In this case, by "a generic metric" we mean that the **generic condition** is satisfied for both timelike and null geodesics. For timelike geodesics, the generic condition

states that every geodesic with tangent vector U^μ must have at least one point on which $R_{\alpha\beta\gamma\delta}U^\alpha U^\delta \neq 0$; for null geodesics, the generic condition states that every geodesic with tangent vector k^μ must have at least one point on which $k_{[\alpha}R_{\beta]\gamma\delta[\epsilon}k_{\zeta]}k^\gamma k^\delta \neq 0$. These fancy conditions simply serve to exclude very special metrics for which the curvature consistently vanishes in some directions.

Singularity theorems exist in many forms, proceeding from various different sets of assumptions. The moral of the story seems to be that typical time-dependent solutions in general relativity usually end in singularities. (Or begin in them; some theorems imply the existence of cosmological singularities, such as the Big Bang.) This represents something of a problem for GR, in the sense that the theory doesn't really apply to the singularities themselves, whose existence therefore represents an incompleteness of description. The traditional attitude toward this issue is to hope that a sought-after quantum theory of gravity will somehow resolve the singularities of classical GR.

In the meantime, we can take solace in the idea that singularities are hidden behind event horizons. This belief is encompassed in the **cosmic censorship conjecture**:

> Naked singularities cannot form in gravitational collapse from generic, initially nonsingular states in an asymptotically flat spacetime obeying the dominant energy condition.

A **naked singularity** is one from which signals can reach $\mathscr{I}^+$; that is, one that is not hidden behind an event horizon. Notice that the conjecture refers to the formation of naked singularities, not their existence; there are certainly solutions in which spacelike naked singularities exist in the past (such as the Schwarzschild white hole) or timelike singularities exist for all times (such as in super-extremal charged black holes, discussed below). The cosmic censorship conjecture has not been proven, although a great deal of effort has gone into finding convincing counterexamples, without success. The requirement that the initial data be in some sense "generic" is important, as numerical experiments have shown that finely-tuned initial conditions are able to give rise to naked singularities. A precise proof of some form of the cosmic censorship conjecture remains one of the outstanding problems of classical general relativity.[3]

A consequence of cosmic censorship (or of certain equivalent assumptions) is that classical black holes never shrink, they only grow bigger. The size of a black hole is measured by the area of the event horizon, by which we mean the spatial area of the intersection of the event horizon with a spacelike slice. We then have Hawking's **area theorem**:

> Assuming the weak energy condition and cosmic censorship, the area of a future event horizon in an asymptotically flat spacetime is nondecreasing.

[3]For a review of cosmic censorship see R.M. Wald, "Gravitational Collapse and Cosmic Censorship," `http://arxiv.org/abs/gr-qc/9710068`.

For a Schwarzschild black hole, the area depends monotonically on the mass, so this theorem implies that Schwarzschild black holes can only increase in mass. But for spinning black holes this is no longer the case; the area depends on a combination of mass and angular momentum, and we will see below that we can actually extract energy from a black hole by decreasing its spin. We can also decrease the mass of a black hole through quantum-mechanical Hawking radiation; this can be traced to the fact that quantum field theory in curved spacetime can violate the weak energy condition.

6.3 ■ KILLING HORIZONS

In the Schwarzschild metric, the Killing vector $K = \partial_t$ goes from being timelike to spacelike at the event horizon. In general, if a Killing vector field χ^μ is null along some null hypersurface Σ, we say that Σ is a **Killing horizon** of χ^μ. Note that the vector field χ^μ will be normal to Σ, since a null surface cannot have two linearly independent null tangent vectors.

The notion of a Killing horizon is logically independent from that of an event horizon, but in spacetimes with time-translation symmetry the two are closely related. Under certain reasonable conditions (made explicit below), we have the following classification:

> Every event horizon Σ in a stationary, asymptotically flat spacetime is a Killing horizon for some Killing vector field χ^μ.
>
> If the spacetime is static, χ^μ will be the Killing vector field $K^\mu = (\partial_t)^\mu$ representing time translations at infinity.
>
> If the spacetime is stationary but not static, it will be axisymmetric with a rotational Killing vector field $R^\mu = (\partial_\phi)^\mu$, and χ^μ will be a linear combination $K^\mu + \Omega_H R^\mu$ for some constant Ω_H.

For example, below we will examine the Kerr metric for spinning black holes, in which the event horizon is a Killing horizon for a linear combination of the Killing vectors for rotations and time translations. In Kerr, the hypersurface on which ∂_t becomes null is actually timelike, so is not a Killing horizon.

Let's be precise about the conditions under which this classification scheme actually holds.[4] Carter has shown that, for static black holes, the event horizon is a Killing horizon for K^μ; this is a purely geometric fact, which holds even without invoking Einstein's equation. In the stationary case, if we assume the existence of a rotational Killing field R^μ with the property that 2-planes spanned by K^μ and R^μ are orthogonal to a family of two-dimensional surfaces, then the event horizon will be a Killing horizon for a linear combination of the two Killing fields, again from purely geometric considerations. If on the other hand we only assume that the black hole is stationary, we cannot prove in general that the event horizon

[4]For a discussion see R. M. Wald, "The thermodynamics of black holes," *Living Rev. Rel.* **4**, 6 (2001), `http://arxiv.org/gr-qc/9912119`.

is axisymmetric. Given Einstein's equation and some conditions on the matter fields, Hawking was able to show that the event horizon of any stationary black hole must be a Killing horizon for some vector field, and furthermore that such horizons must either be stationary or axisymmetric. For the rest of this chapter we will speak as if the above classification holds; however, making assumptions about matter fields is notoriously tricky, and we should keep in mind the possibility in principle of finding black holes that are not static or axisymmetric, for which the event horizon might not be a Killing horizon.

It's important to point out that, while event horizons for stationary asymptotically flat spacetimes will typically be Killing horizons, it's easy to have Killing horizons that have nothing to do with event horizons. Consider Minkowski space in inertial coordinates, $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$; clearly there are no event horizons in this spacetime. The Killing vector that generates boosts in the x-direction is

$$\chi = x\partial_t + t\partial_x, \tag{6.5}$$

with norm

$$\chi_\mu \chi^\mu = -x^2 + t^2. \tag{6.6}$$

This goes null at the null surfaces

$$x = \pm t, \tag{6.7}$$

which are therefore Killing horizons. By combining the boost Killing vector with translational and rotational Killing vectors, we can move these horizons through the manifold; there are Killing horizons all over. In more interesting spacetimes, of course, there will be fewer Killing vector fields, and the associated horizons (if any) will have greater physical significance.

To every Killing horizon we can associate a quantity called the **surface gravity**. Consider a Killing vector χ^μ with Killing horizon Σ. Because χ^μ is a normal vector to Σ, along the Killing horizon it obeys the geodesic equation,

$$\chi^\mu \nabla_\mu \chi^\nu = -\kappa \chi^\nu, \tag{6.8}$$

where the right-hand side arises because the integral curves of χ^μ may not be affinely parameterized. The parameter κ is the surface gravity; it will be constant over the horizon, except for a "bifurcation two-sphere" where the Killing vector vanishes and κ can change sign. (This happens, for example, at the center of the Kruskal diagram in the Schwarzschild solution.) Using Killing's equation $\nabla_{(\mu}\chi_{\nu)} = 0$ and the fact that $\chi_{[\mu}\nabla_\nu \chi_{\sigma]} = 0$ (since χ^μ is normal to Σ), it is straightforward to derive a nice formula for the surface gravity:

$$\kappa^2 = -\tfrac{1}{2}(\nabla_\mu \chi_\nu)(\nabla^\mu \chi^\nu). \tag{6.9}$$

The expression on the right-hand side is to be evaluated at the horizon Σ. You are encouraged to check this formula yourself.

The surface gravity associated with a Killing horizon is in principle arbitrary, since we can always scale a Killing field by a real constant and obtain another Killing field. In a static, asymptotically flat spacetime, the time-translation Killing vector $K = \partial_t$ can be normalized by setting

$$K_\mu K^\mu (r \to \infty) = -1. \tag{6.10}$$

This in turn fixes the surface gravity of any associated Killing horizon. If we are in a stationary spacetime, where the Killing horizon is associated with a linear combination of time translations and rotations, fixing the normalization of $K = \partial_t$ also fixes this linear combination, so the surface gravity remains unique.

The reason why κ is called the "surface gravity" becomes clear only when the spacetime is static. In that case we have the following interpretation:

> In a static, asymptotically flat spacetime, the surface gravity is the acceleration of a static observer near the horizon, as measured by a static observer at infinity.

To make sense of such a statement, let's first consider static observers. By a static observer we mean one whose four-velocity U^μ is proportional to the time-translation Killing field K^μ:

$$K^\mu = V(x) U^\mu. \tag{6.11}$$

Since the four-velocity is normalized to $U_\mu U^\mu = -1$, the function V is simply the magnitude of the Killing field,

$$V = \sqrt{-K_\mu K^\mu}, \tag{6.12}$$

and hence ranges from zero at the Killing horizon to unity at infinity. V is sometimes called the "redshift factor," since it relates the emitted and observed frequencies of a photon as measured by static observers. Recall that the conserved energy of a photon with four-momentum p^μ is $E = -p_\mu K^\mu$, while the frequency measured by an observer with four-velocity U^μ will be $\omega = -p_\mu U^\mu$. Therefore

$$\omega = \frac{E}{V}, \tag{6.13}$$

and a photon emitted by static observer 1 will be observed by static observer 2 to have wavelength $\lambda = 2\pi/\omega$ given by

$$\lambda_2 = \frac{V_2}{V_1} \lambda_1. \tag{6.14}$$

In particular, at infinity where $V = 1$, we will observe a wavelength $\lambda_\infty = \lambda_1 / V_1$.

Now we turn to the idea of "acceleration as viewed from infinity." A static observer will not typically be moving on a geodesic; for example, particles tend to fall into black holes rather than hovering next to them at fixed spatial coordinates.

We can express the four-acceleration $a^\mu = U^\sigma \nabla_\sigma U^\mu$ in terms of the redshift factor as

$$a_\mu = \nabla_\mu \ln V, \tag{6.15}$$

as you can easily check. The magnitude of the acceleration,

$$a = \sqrt{a_\mu a^\mu} = V^{-1}\sqrt{\nabla_\mu V \nabla^\mu V}, \tag{6.16}$$

will go to infinity at the Killing horizon—it will take an infinite acceleration to keep an object on a static trajectory. But an observer at infinity will detect the acceleration to be "redshifted" by a factor V; this turns out to be the surface gravity. Thus, we claim that

$$\kappa = Va = \sqrt{\nabla_\mu V \nabla^\mu V}, \tag{6.17}$$

evaluated at the horizon Σ. You can check that this expression agrees with (6.9). The surface gravity is the product of zero (V) and infinity (a), but will typically be finite. When we say that the observed acceleration is redshifted, we have in mind stretching a test string from a static object at the horizon to an observer at infinity, and measuring the acceleration on the end of the string at infinity. (It is worth taking the time to see if you can promote this hand-waving argument to something more rigorous.)

What goes wrong with the above considerations if the spacetime is stationary but not static? We still have an asymptotically time-translation Killing vector $K = \partial_t$, and we can define stationary observers as ones whose four-velocities are parallel to K^μ, as in (6.11); the redshift will continue to be given by (6.14). The problem is that K^μ won't become null at a Killing horizon, but generally at some timelike surface outside the horizon. This place where $K^\mu K_\mu = 0$ is called the **stationary limit surface** (or sometimes "ergosurface"), since inside this surface K^μ is spacelike, and consequently no observer can remain stationary, even if it is still outside the event horizon. Such an observer has to move with respect to the Killing field, but need not move in the direction of the black hole. From (6.12) and (6.14), the redshift of a stationary observer diverges as we approach the stationary limit surface, which is therefore also called the **infinite redshift surface**. As we will see in our discussion of the Kerr metric, the region between the stationary limit surface and the event horizon, the ergosphere, is a place where timelike paths are inevitably dragged along with the rotation of the black hole. We will continue to use "surface gravity" as a label in stationary spacetimes, which we will calculate using the Killing vector χ^μ, which actually does go null on the event horizon, even if the resulting quantity cannot be interpreted as the gravitational acceleration of a stationary observer as seen at infinity.

Let's apply these notions to Schwarzschild to see how they work. For the metric

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) \mathrm{d}t^2 + \left(1 - \frac{2GM}{r}\right)^{-1} \mathrm{d}r^2 + r^2 d\Omega^2, \tag{6.18}$$